Calculation of the Feynman diagram in the main text

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I. TWO-PHOTON EXCHANGE FEYNMAN DIAGRAM

Consider an electron moving at a distance r above a Chern-Simon (CS) surface at $z = 0$. The action has the following form

$$S = \int d^4x \, \left\{ \bar{\psi} \left[\gamma^\mu (p_\mu - eA_\mu) - m \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} + \int d^4x \, \epsilon^{\alpha\beta\gamma 3} A_\alpha \partial_\beta A_\gamma \, \delta(x_3). \tag{1}$$

We separate the whole action into free part and interaction part, i.e., $S = S_0 + S_I$, where

$$S_0 = \int d^4x \, \left\{ \bar{\psi} \left[\gamma^\mu p_\mu - m \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}; \tag{2}$$

$$S_I = S_I^a + S_I^b = \int d^4x \, \bar{\psi} \left(-e\gamma^\mu A_\mu \right) \psi + \int d^4x \, \epsilon^{\alpha\beta\gamma 3} A_\alpha \partial_\beta A_\gamma \, \delta(x_3). \tag{3}$$

Note that S_I^a and S_I^b, respectively, represent electron-photon vortex and CS vortex.

Now, we can consider the generating function

$$Z = \frac{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0 + iS_I}}{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0}} = \frac{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0} \left[1 + iS_I + \frac{1}{2}(iS_I)^2 + \frac{1}{3!}(iS_I)^3 + ... \right]}{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0}} \tag{4}$$

So the lowest order contribution from the CS plate is a two-photon process: (two electron-photon vortices and one CS vortex)

$$Z = \frac{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0} \left[\frac{1}{2} (i \, S_I^a)^2 (i \, S_I^b) \right]}{\int D[\bar{\psi}, \psi] D[A] \; e^{iS_0}}. \tag{5}$$

The relevant Feynman diagram [see the Figure 1] describe the interaction between the electron and Chern-Simon term can be calculated via

$$M = \int d^4x \int d^4w \int d^4z \; \bar{\psi}(z) \, (-ie\gamma^\mu) D_{\mu\alpha}(z - x) \, G(z - w) \, (i\partial_\beta) \, D_{\rho\delta}(x - w) \, \delta(x_3) \epsilon^{\alpha\beta\rho 3} \, (-ie\gamma^\delta) \, \psi(w) \tag{6}$$

where G and D correspond to Feynman propagators of electron and photon, respectively.

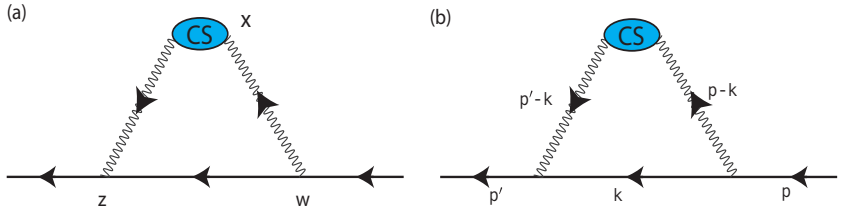


Figure 1: The Feynman diagrams in real space (a) and in momentum space (b).

Substitute the Fourier transformation of the Feynman propagators

$$D_{\mu\alpha}(z-x) = \int \frac{d^4k'}{(2\pi)^4}\,\frac{(-i)g_{\mu\alpha}}{k'^2+i\epsilon}\,e^{-ik'(z-x)} \tag{7}$$

$$D_{\rho\delta}(x-w) = \int \frac{d^4k''}{(2\pi)^4}\,\frac{(-i)g_{\rho\delta}}{k''^2+i\epsilon}\,e^{-ik''(x-w)} \tag{8}$$

$$G(z-w) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{\gamma^\mu k_\mu - m + i\epsilon}e^{-ik(z-w)} \tag{9}$$

into the above expression Eqn. (6), and one can obtain:

$$\begin{aligned}
M =& \int d^4x \int d^4w \int d^4z\ \delta(x_3)\times\bar{u}(p')e^{ip'z}(ie\gamma^\mu)\\
&\int \frac{d^4k'}{(2\pi)^4}D_{\mu\alpha}(k')e^{-ik'(z-x)}\times\ (i\partial_\beta)\int \frac{d^4k''}{(2\pi)^4}D_{\rho\delta}(k'')e^{-ik''(x-w)}\epsilon^{\alpha\beta\rho3}\times\int \frac{d^4k}{(2\pi)^4}G(k)e^{-ik(z-w)}\ e^{-ipw}(ie\gamma^\delta)u(p)\\
=&\bar{u}(p')(ie\gamma^\mu)\int dx_0dx_1dx_2dx_3\int dw_0dw_1dw_2dw_3\int dz_0dz_1dz_2dz_3\ \delta(x_3)\times\\
&\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4}\int \frac{d^4k''}{(2\pi)^4}\left[D_{\mu\alpha}(k')\times(k''_\beta)\times D_{\rho\delta}(k'')\right]\epsilon^{\alpha\beta\rho3}G(k)(ie\gamma^\delta)e^{i(k'-k'')x}e^{i(k''+k-p)w}e^{-i(k+k'-p')z}u(p)\\
=&\frac{1}{2\pi}\bar{u}(p')(ie\gamma^\mu)\int d^4k\int d^4k'\int d^4k''\left[D_{\mu\alpha}(k')\times(k''_\beta)\times D_{\rho\delta}(k'')\right]\epsilon^{\alpha\beta\rho3}G(k)(ie\gamma^\delta)\times\\
&\delta(k'-k'')_{0,1,2}\ \delta(k'+k-p')\ \delta(k''+k-p)u(p)\\
=&\frac{1}{2\pi}\,\bar{u}(p')\,(ie\gamma^\mu)\int d^4k\,\delta(p'-p)_{0,1,2}\,[D_{\mu\alpha}(p'-k)\times(p-k)_\beta\times D_{\rho\delta}(p-k)]\,\epsilon^{\alpha\beta\rho3}G(k)(ie\gamma^\delta)\times\ u(p)
\end{aligned} \tag{10}$$

In the Feynman gauge, photon's propagator is diagonal. So the scattering amplitude is

$$\begin{aligned}
M =&\frac{1}{2\pi}\,\delta(p'-p)_{0,1,2}\ \bar{u}(p')(ie\gamma^\mu)\int d^4k\times\\
&\frac{(-i)g_{\mu\mu}}{(p'-k)^2}\times\frac{(-i)g_{\rho\rho}}{(p-k)^2}\times(p-k)_\beta\times\frac{i}{\gamma^\nu k_\nu - m}\,\epsilon^{\mu\beta\rho3}(ie\gamma^\rho)\,u(p)
\end{aligned} \tag{11}$$

We can explicitly write down all possible terms according to the value of β in the above formula.
(i) $\beta = 1$ term:

$$\begin{aligned}
M_1 =& -\frac{i}{2\pi}\,\delta(p'-p)_{0,1,2}\,\bar{u}(p')\,(e^2\gamma^0)\int d^4k\times\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{(p-k)_1}{\gamma^\nu k_\nu - m}\gamma^2\times\ u(p)\\
&+\frac{i}{2\pi}\,\delta(p'-p)_{0,1,2}\,\bar{u}(p')\,(e^2\gamma^2)\int d^4k\times\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{(p-k)_1}{\gamma^\nu k_\nu - m}\gamma^0\times\ u(p)\\
=&-\frac{ie^2}{2\pi}\,\delta(p'-p)_{0,1,2}\,\bar{u}(p')\times\\
&\int d^4k\,\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{\gamma^0(p-k)_1\,(\gamma^\nu k_\nu+m)\gamma^2-\gamma^2(p-k)_1\,(\gamma^\nu k_\nu+m)\gamma^0}{k^2-m^2}\times\ u(p)
\end{aligned} \tag{12}$$

(ii) $\beta = 2$ term:

$$\begin{aligned}
M_2 =&\frac{ie^2}{2\pi}\,\delta(p-p)_{0,1,2}\,\bar{u}(p')\times\\
&\int d^4k\,\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{\gamma^0(p-k)_2\,(\gamma^\nu k_\nu+m)\gamma^1-\gamma^1(p-k)_2\,(\gamma^\nu k_\nu+m)\gamma^0}{k^2-m^2}\times\ u(p)
\end{aligned} \tag{13}$$

(iii) $\beta = 0$ term:

$$M_3 = -\frac{ie^2}{2\pi}\,\delta(p'-p)_{0,1,2}\,\bar{u}(p')\times \int d^4k\,\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{\gamma^1(p-k)_0(\gamma^\nu k_\nu+m)\gamma^2-\gamma^2(p-k)_0(\gamma^\nu k_\nu+m)\gamma^1}{k^2-m^2}\times u(p) \tag{14}$$

II. CALCULATION OF THE INTEGRALS

First of all, let's perform Feynman parametrization to simplify the denominator.
Using Feynman parametrization trick $\frac{1}{ABC} = 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{[u_2A+(u_1-u_2)B+(1-u_1)C]^3}$, one can obtain

$$\begin{aligned}\frac{1}{(p-k)^2}\times\frac{1}{(p'-k)^2}\times\frac{1}{k^2-m^2} &= 2\int_0^1 du_1\int_0^{u_1} du_2 \frac{1}{\left[u_2(p-k)^2+(u_1-u_2)(p'-k)^2+(1-u_1)(k^2-m^2)\right]^3}\\ &= 2\int_0^1 du_1\int_0^{u_1} du_2\frac{1}{D^3}\end{aligned} \tag{15}$$

Here,

$$\begin{aligned}D =&u_2(p-k)^2+(u_1-u_2)(p'-k)^2+(1-u_1)(k^2-m^2)\\ =&u_2\left[(p-k)_0^2-(p-k)_1^2-(p-k)_2^2-(p-k)_3^2\right]+(u_1-u_2)\left[(p'-k)_0^2-(p'-k)_1^2-(p'-k)_2^2-(p'-k)_3^2\right]\\ &+(1-u_1)\left(k_0^2-k_1^2-k_2^2-k_3^2-m^2\right)\\ =&-u_2(p_3-k_3)^2+u_2(p_3'-k_3)^2+u_1\left[(p'-k)_0^2-(p'-k)_1^2-(p'-k)_2^2-(p'-k)_3^2\right]\\ &+(1-u_1)\left(k_0^2-k_1^2-k_2^2-k_3^2-m^2\right)\\ =&2u_2(p_3-p_3')k_3+u_1\left[({p_0'}^2-2p_0'k_0)-({p_1'}^2-2p_1'k_1)-({p_2'}^2-2p_2'k_2)-({p_3'}^2-2p_3'k_3)\right]\\ &+\left(k_0^2-k_1^2-k_2^2-k_3^2\right)-(1-u_1)m^2\\ =&(k_0^2-2u_1p_0'k_0)-(k_1^2-2u_1p_1'k_1)-(k_2^2-2u_1p_2'k_2)-\left[k_3^2-2u_1p_3'k_3+2u_2(p_3'-p_3)k_3\right]\\ &+u_1({p_0'}^2-{p_1'}^2-{p_2'}^2-{p_3'}^2)-(1-u_1)m^2\\ =&(k_0^2-2u_1p_0'k_0)-(k_1^2-2u_1p_1'k_1)-(k_2^2-2u_1p_2'k_2)-\left[k_3^2-2u_1p_3'k_3+2u_2(p_3'-p_3)k_3\right]\\ &+u_1m^2-(1-u_1)m^2\\ =&(k_0-u_1p_0')^2-(k_1-u_1p_1')^2-(k_2-u_1p_2')^2-\left[k_3-u_1p_3'+u_2(p_3'-p_3)\right]^2\\ &+(2u_1-1)m^2-(u_1p_0')^2+(u_1p_1')^2+(u_1p_2')^2+\left[-u_1p_3'+u_2(p_3'-p_3)\right]^2\\ =&(k_0-u_1p_0')^2-(k_1-u_1p_1')^2-(k_2-u_1p_2')^2-\left[k_3-u_1p_3'+u_2(p_3'-p_3)\right]^2\\ &+(-u_1^2+2u_1-1)m^2-u_1^2{p_3'}^2+\left((u_1-u_2)p_3'+u_2p_3\right)^2\\ =&l_0^2-l_1^2-l_2^2-l_3^2-(1-u_1)^2m^2-u_1^2{p_3'}^2+\left[-u_1p_3'+u_2(p_3'-p_3)\right]^2\\ =&l_0^2-l_1^2-l_2^2-l_3^2-T^2\end{aligned} \tag{16}$$

where $T^2 = (1-u_1)^2m^2+u_1^2{p_3'}^2-\left[u_1p_3'-u_2(p_3'-p_3)\right]^2$. We have used substitution of variables $l_0 = k_0-u_1p_0'$, $l_1 = k_1-u_1p_1'$, $l_2 = k_2-u_1p_2'$, $l_3 = k_3-u_1p_3'+u_2(p_3'-p_3)$, and on-shell condition of the external legs of electrons in Eqn. (16).

Second, let's make some simplification of the numerator.

The numerator in M_1 is

$$\begin{aligned}&\gamma^0(p-k)_1\left(\gamma^\nu k_\nu+m\right)\gamma^2-\gamma^2(p-k)_1\left(\gamma^\nu k_\nu+m\right)\gamma^0\\=&\gamma^0\gamma^2(p_1-k_1)\left[-\gamma^2(\not{k}+m)\gamma^2+\gamma^0(\not{k}+m)\gamma^0\right]\\=&\gamma^0\gamma^1\gamma^2(\not{p}_1-\not{k}_1)\left[-\gamma^2(\not{k}+m)\gamma^2+\gamma^0(\not{k}+m)\gamma^0\right]\\=&2\gamma^0\gamma^1\gamma^2(\not{p}_1-\not{k}_1)\left[m-\not{k}_1-\not{k}_3\right]\end{aligned}\tag{17}$$

The numerator in M_2 is

$$\begin{aligned}&-\left[\gamma^0(p-k)_2\left(\gamma^\nu k_\nu+m\right)\gamma^1-\gamma^1(p-k)_2\left(\gamma^\nu k_\nu+m\right)\gamma^0\right]\\=&-\gamma^0\gamma^1\gamma^2(\not{p}_2-\not{k}_2)\left[\gamma^1(\not{k}+m)\gamma^1-\gamma^0(\not{k}+m)\gamma^0\right]\\=&2\gamma^0\gamma^1\gamma^2(\not{p}_2-\not{k}_2)\left[m-\not{k}_2-\not{k}_3\right]\end{aligned}\tag{18}$$

The numerator in M_3 is

$$\begin{aligned}&\gamma^1(p-k)_0(\gamma^\nu k_\nu+m)\gamma^2-\gamma^2(p-k)_0(\gamma^\nu k_\nu+m)\gamma^1\\=&-\gamma^0\gamma^1\gamma^2(\not{p}_0-\not{k}_0)\left[\gamma^2(\not{k}+m)\gamma^2+\gamma^1(\not{k}+m)\gamma^1\right]\\=&2\gamma^0\gamma^1\gamma^2(\not{p}_0-\not{k}_0)\left[m-\not{k}_0-\not{k}_3\right]\end{aligned}\tag{19}$$

If we add up M_1, M_2, M_3, the total numerator is

$$\begin{aligned}&2m\gamma^0\gamma^1\gamma^2\left[(\not{p}_0-\not{k}_0)+(\not{p}_1-\not{k}_1)+(\not{p}_2-\not{k}_2)\right]\\&-2\gamma^0\gamma^1\gamma^2\left[(\not{p}_0-\not{k}_0)(\not{k}_0+\not{k}_3)+(\not{p}_1-\not{k}_1)(\not{k}_1+\not{k}_3)+(\not{p}_2-\not{k}_2)(\not{k}_2+\not{k}_3)\right]\end{aligned}\tag{20}$$

Next, we regroup the numerator into four parts.

a.

$$t_1=2m\gamma^0\gamma^1\gamma^2\left[(\not{p}_0-\not{k}_0)+(\not{p}_1-\not{k}_1)+(\not{p}_2-\not{k}_2)\right]\tag{21}$$

b.

$$t_2=2\gamma^0\gamma^1\gamma^2(k_0^2-k_1^2-k_2^2)\tag{22}$$

c.

$$t_3=-2\gamma^0\gamma^1\gamma^2\left[(\not{p}_0+\not{p}_1+\not{p}_2)\not{k}_3-(\not{k}_0+\not{k}_1+\not{k}_2)\not{k}_3\right]\tag{23}$$

d.

$$t_4=-2\gamma^0\gamma^1\gamma^2\left[p_0k_0-p_1k_1-p_2k_2\right]\tag{24}$$

The integral that we need to calculate becomes

$$M = -\frac{ie^2}{2\pi}\delta(p'-p)_{0,1,2} \times 2\int_0^1 du_1 \int_0^{u_1} du_2 \int d^4l \frac{t_1+t_2+t_3+t_4}{[l_0^2-l_1^2-l_2^2-l_3^3-T^2]^3}, \tag{25}$$

where $T^2 = \alpha^2 - [u_1 p_3' - u_2(p_3' - p_3)]^2$ with $\alpha^2 = (1-u_1)^2 m^2 + u_1^2 {p_3'}^2$.

In the following, we will often use the typical integral of momentum:

$$\begin{aligned}
&\int d^4l \frac{1}{(l_0^2-l_1^2-l_2^2-l_3^2-T^2+i\epsilon)^3} \text{ (Wick rotation: } l_0 \to il_0) \\
&= -i\int d^4l \frac{1}{(l_0^2+l_1^2+l_2^2+l_3^2+T^2)^3} \\
&= -i2\pi^2 \int_0^\infty dl \frac{l^3}{(l^2+T^2)^3} = -i\frac{\pi^2}{2}\frac{1}{T^2}
\end{aligned} \tag{26}$$

We will often use variable substitution $l_0 = k_0 - u_1 p_0'$, $l_1 = k_1 - u_1 p_1'$, $l_2 = k_2 - u_1 p_2'$, $l_3 = k_3 - u_1 p_3' + u_2(p_3' - p_3)$; and then perform Wick rotation $l_0 \mapsto il_0$ in the following context. In addition, we use the relations $p_0' = p_0$, $p_1' = p_1$, $p_2' = p_2$ due to the $\delta(p'-p)_{0,1,2}$ function in M. Now, we can calculate the following terms based on four different types of numerators.

a.

$$\begin{aligned}
t_1 =& 2m\gamma^0\gamma^1\gamma^2 \left[(\not{p}_0' - \not{k}_0) + (\not{p}_1' - \not{k}_1) + (\not{p}_2' - \not{k}_2)\right] \\
&= -2m\gamma^0\gamma^1\gamma^2 \left[(\not{l}_0 + u_1\not{p}_0' - \not{p}_0') + (\not{l}_1 + u_1\not{p}_1' - \not{p}_1') + (\not{l}_2 + u_1\not{p}_2' - \not{p}_2')\right] \\
&= 2m\gamma^0\gamma^1\gamma^2(1-u_1)\left[\not{p}_0' + \not{p}_1' + \not{p}_2'\right] \text{ (dropped odd power of } l_\mu) \\
&= 2m(1-u_1)\left[\not{p}_0' + \not{p}_1' + \not{p}_2'\right]\gamma^0\gamma^1\gamma^2 \text{ (removed } \not{p}_\mu' \text{ to the front)} \\
&= 2m(1-u_1)\left[m - \not{p}_3'\right]\gamma^0\gamma^1\gamma^2 \text{ (used on-shell condition of Dirac equation } \bar{u}(p')(\not{p}' - m) = 0) \\
&\approx 2m^2(1-u_1)\gamma^0\gamma^1\gamma^2 \text{ (in nonrelativistic limit } m >> p_1, p_2, p_3)
\end{aligned} \tag{27}$$

b.

$$\begin{aligned}
t_2 =& 2\gamma^0\gamma^1\gamma^2 \left[(l_0 + u_1 p_0')^2 - (l_1 + u_1 p_1')^2 - (l_2 + u_1 p_2')^2\right] \\
&= 2\gamma^0\gamma^1\gamma^2 \left[l_0^2 - l_1^2 - l_2^2 + u_1^2({p_0'}^2 - {p_1'}^2 - {p_2'}^2)\right] \text{ (dropped odd power of } l_\mu \text{, and Wick rotation } \to) \\
&= 2\gamma^0\gamma^1\gamma^2 \left[-l_0^2 - l_1^2 - l_2^2 + u_1^2({p_0'}^2 - {p_1'}^2 - {p_2'}^2)\right] \\
&= -2\gamma^0\gamma^1\gamma^2(l_0^2 + l_1^2 + l_2^2) + 2\gamma^0\gamma^1\gamma^2 u_1^2 \left({p_0'}^2 - {p_1'}^2 - {p_2'}^2\right) \\
&= -2\gamma^0\gamma^1\gamma^2(l_0^2 + l_1^2 + l_2^2) + 2\gamma^0\gamma^1\gamma^2 u_1^2 \left(m^2 + {p_3'}^2\right)
\end{aligned} \tag{28}$$

The first term $2\gamma^0\gamma^1\gamma^2(-l_0^2+l_1^2+l_2^2)$ contributes to the total scattering amplitude as

$$\begin{aligned}
&-\frac{ie^2}{2\pi}\delta(p'-p)_{0,1,2}\times 2\int_0^1 du_1\int_0^{u_1} du_2(-i)\int d^4l\frac{-2\gamma^0\gamma^1\gamma^2(l_0^2+l_1^2+l_2^2)}{\left(l_0^2+l_1^2+l_2^2+l_3^2+T^2\right)^3}\\
&=\frac{e^2}{\pi}\delta(p'-p)_{0,1,2}\times 2\gamma^0\gamma^1\gamma^2\int_0^1 du_1\int_0^{u_1} du_2\times\frac{3}{4}\int_0^\infty dl\,(2\pi^2)\frac{l^5}{\left(l^2+T^2\right)^3}\\
&=e^2\pi\delta(p'-p)_{0,1,2}\times\gamma^0\gamma^1\gamma^2\int_0^1 du_1\int_0^{u_1} du_2\times\frac{3}{2}\Gamma(0)\\
&=\frac{3\pi e^2}{4}\delta(p'-p)_{0,1,2}\gamma^0\gamma^1\gamma^2\Gamma(0)
\end{aligned}\tag{29}$$

This term is independent of scattering momentum, thus does not contribute to the effective potential.

Now, we can consider the second term

$$\begin{aligned}
&2\gamma^0\gamma^1\gamma^2u_1^2\left(m^2+{p_3'}^2\right)\ \text{(in non-relativistic limit)}\\
&\approx 2\gamma^0\gamma^1\gamma^2u_1^2m^2
\end{aligned}\tag{30}$$

c.

$$\begin{aligned}
t_3&=2\gamma^0\gamma^1\gamma^2\left[(\not k_0-\not p_0')+(\not k_1-\not p_1')+(\not k_2-\not p_2')\right]\not k_3\\
&=2\gamma^0\gamma^1\gamma^2\left[(\not l_0+u_1\not p_0'-\not p_0')+(\not l_1+u_1\not p_1'-\not p_1')+(\not l_2+u_1\not p_2'-\not p_2')\right]\times\left[\not l_3+u_1\not p_3'-u_2(\not p_3'-\not p_3)\right]\\
&=-2\gamma^0\gamma^1\gamma^2(1-u_1)u_1\left(\not p_0'+\not p_1'+\not p_2'\right)\not p_3'+2\gamma^0\gamma^1\gamma^2(1-u_1)u_2\left(\not p_0'+\not p_1'+\not p_2'\right)(\not p_3'-\not p_3)\\
&=-2(1-u_1)u_1\left[\gamma^1\gamma^2\gamma^3p_0'p_3'+\gamma^0\gamma^2\gamma^3p_1'p_3'-\gamma^0\gamma^1\gamma^3p_2'p_3'\right]\\
&\quad+2(1-u_1)u_2\left[\gamma^1\gamma^2\gamma^3p_0'(p_3'-p_3)+\gamma^0\gamma^2\gamma^3p_1'(p_3'-p_3)-\gamma^0\gamma^1\gamma^3p_2'(p_3'-p_3)\right]\\
&=-2(1-u_1)u_1\left[\gamma^0\gamma^2\gamma^3p_1'p_3'-\gamma^0\gamma^1\gamma^3p_2'p_3'\right]+2(1-u_1)u_2\left[\gamma^0\gamma^2\gamma^3p_1'(p_3'-p_3)-\gamma^0\gamma^1\gamma^3p_2'(p_3'-p_3)\right]\\
&=-2(1-u_1)u_1A+2(1-u_1)u_2B
\end{aligned}\tag{31}$$

If we want to calculate $\int_0^1 du_1\int_0^{u_1}du_2\,\frac{t_3}{T^2}$, we need to perform the following two integrals:

$$\int_0^{u_1}du_2\frac{u_2}{\alpha^2-\left[u_1p_3'-u_2(p_3'-p_3)\right]^2}\ \ [2(1-u_1)B]\tag{32}$$

$$\int_0^{u_1}du_2\frac{1}{\alpha^2-\left[u_1p_3'-u_2(p_3'-p_3)\right]^2}\ \ [-2(1-u_1)u_1A]\tag{33}$$

where $\alpha^2=(1-u_1)^2m^2+u_1^2{p_3'}^2$.

Because $\int_0^{u_1}du_2\frac{u_2}{\alpha^2-\left[u_1p_3'-u_2(p_3'-p_3)\right]^2}=\frac{u_1^2}{\alpha^2}\cdot\frac{p_3'}{p_3'-p_3}$, and $\int_0^{u_1}du_2\frac{1}{\alpha^2-\left[u_1p_3'-u_2(p_3'-p_3)\right]^2}=\frac{u_1}{\alpha^2}$. Then, you will find these two integrals exactly canceled with each other. So we don't need to consider the term c anymore. Note that these integrals are calculate under the assumption $p_1\,,p_2\,,p_3<<m$.

There is another way to prove the vanish of t_3 term by invoking the symmetry of the integral. With the substitution $u_2\to u_1-u_2$, the whole integral remains unchanged. So one can use the combination $\frac{1}{2}\left(u_2+u_1-u_2\right)=\frac{u_1}{2}$ to represent u_2. Remember $p_3'=-p_3$, then one can show the two terms in t_3 cancel out with each other.

d.

$$\begin{aligned} t_4 =& -2\gamma^0\gamma^1\gamma^2\left[p_0' k_0 - p_1' k_1 - p_2' k_2\right] \\ =& -2\gamma^0\gamma^1\gamma^2\left[p_0'(l_0+u_1 p_0') - p_1'(l_1+u_1 p_1') - p_2'(l_2+u_1 p_2')\right] \\ =& -2\gamma^0\gamma^1\gamma^2 u_1\left({p_0'}^2 - {p_1'}^2 - {p_2'}^2\right) \\ =& -2\gamma^0\gamma^1\gamma^2 u_1\left(m^2 + {p_3'}^2\right) \\ \approx& -2\gamma^0\gamma^1\gamma^2 u_1 m^2 \end{aligned} \tag{34}$$

Add up t_1, t_2 and t_4, and the numerator becomes

$$t_1 + t_2 + t_4 = 2m^2(1-u_1+u_1^2-u_1)\gamma^0\gamma^1\gamma^2 = 2m^2(1-u_1)^2\gamma^0\gamma^1\gamma^2 \tag{35}$$

Consider $t_1 + t_2 + t_4$, we need to calculate the integral:

$$\begin{aligned} &\left[2m^2\gamma^0\gamma^1\gamma^2\right]\int_0^1 du_1\int_0^{u_1} du_2(-\frac{i\pi^2}{2})\frac{(1-u_1)^2}{T^2} \\ &= \left[2m^2\gamma^0\gamma^1\gamma^2\right]\left(-\frac{i\pi^2}{2}\right)\int_0^1 du_1\frac{u_1(1-u_1)^2}{\alpha^2} \\ &= \left[(-i\pi^2)\gamma^0\gamma^1\gamma^2\right]\frac{m^4 - \pi m^3 p_3' - 4m^2{p_3'}^2 + 3m^2{p_3'}^2\, log(\frac{m}{p_3'})^2}{2m^4} \end{aligned} \tag{36}$$

We collect all the gradients and only care about the off-diagonal scattering amplitude, which is

$$\begin{aligned} M &= \delta(p'-p)_{0,1,2}\,\bar{u}\left(-\frac{ie^2}{2\pi}\right)\times\left[(-i\pi^2)\gamma^0\gamma^1\gamma^2\right]\left(-\frac{\pi}{2}\right)\frac{p_3'}{m}\times u \\ &= \delta(p'-p)_{0,1,2}\,\bar{u}\left(\frac{\pi^2e^2}{4}\right)\gamma^0\gamma^1\gamma^2\frac{p_3'}{m}\times u \end{aligned} \tag{37}$$

In non-relativistic limit, $u \to \left(\xi,\ \frac{\mathbf{p}\cdot\sigma}{2m}\xi\right)^T$, to the first order p_3'/m, the scattering amplitude for spin up/down electron is

$$M = \delta(p'-p)_{0,1,2}\; i\,\xi^\dagger\left(-\frac{\pi^2e^2}{4}\right)\frac{p_3'}{m}\sigma_3\,\xi \tag{38}$$

In the scattering process, the transferred momentum is $\tilde{p} = (0,0,0,2p_3')$. Fourier transform the scattering matrix, we can get the effective interaction

$$\begin{aligned} V(r) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tilde{p}_0\, d\tilde{p}_1\, d\tilde{p}_2\;\delta(\tilde{p})_{0,1,2}\; e^{-i\tilde{p}_0x_0+i\tilde{p}_1x_1+i\tilde{p}_2x_2}\;\times 2\int_0^{\infty} d\tilde{p}_3\;\left(-\frac{\pi^2e^2}{4}\right)\frac{p_3'}{m}\sigma_3 e^{i\tilde{p}_3 r} \\ &= \frac{1}{16\pi^4}\left(-\frac{\pi^2e^2}{4}\right)\times 4\int_0^{\infty} dp_3'\,\frac{p_3'}{m}\,\sigma_3\, e^{2ip_3'r} \\ &= -\frac{e^2}{16\pi^2}\int_0^{\infty} dp_3'\,\frac{p_3'}{m}\,\sigma_3\, e^{2ip_3'r}\ \text{(Wick rotation } p_3' \to ip_3') \\ &= \frac{e^2}{16\pi^2}\int_0^{\infty} dp_3'\,\frac{p_3'}{m}\,\sigma_3\, e^{-2p_3'r} \\ &= \frac{e^2}{64\pi^2}\frac{1}{mr^2}\sigma_3 \end{aligned} \tag{39}$$

Note that, in our calculation, the coefficient before Chern-Simon term is 1. For the surface of a topological insulator,

we can put the coefficient as $\frac{\kappa}{2} = \frac{j\alpha}{2}$, where j is an odd number. Thus, our final result becomes

$$\begin{aligned} V(r) &= j\frac{\kappa}{2} \times \frac{e^2}{64\pi^2}\frac{1}{mr^2}\sigma_3 \\ &= j\frac{\alpha^2}{32\pi}\frac{1}{mr^2}\sigma_3 \end{aligned} \tag{40}$$