Realization of "Time Crystal" Lagrangians and Emergent Sisyphus Dynamics

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We demonstrate how non-convex "time crystal" Lagrangians arise in the effective description of conventional, realizable physical systems. Such embeddings allow for the resolution of dynamical singularities that arise in the reduced description. Sisyphus dynamics, featuring intervals of forward motion interrupted by quick resets, is a generic consequence. Near the would-be singularity of the time crystal, we find striking microstructure.

The concept of "time crystal" [1–3] has attracted great interest recently, both theoretical [4–12] and experimental [13–15]. Most of the recent activity concerns many-body physics, and the possibility to break time translation symmetry in ways that retain as much as possible (which turns out to be a lot) of the structure which physicists have come to associate with spontaneous symmetry breaking in other contexts, such as sharp phase transitions, order parameters, and generalized rigidity. Here we will explore a different but complementary aspect: effective dynamics.

To be concrete, let us consider adding a potential to the minimal time crystal Lagrangian for a single degree of freedom [1]:

$$L \;=\; \frac{1}{12}\dot{y}^4 - \frac{1}{2}\dot{y}^2 - V(y). \tag{1}$$

This leads to the energy

$$E = \frac{1}{4}(\dot{y}^2 - 1)^2 + V(y) - \frac{1}{4}. \tag{2}$$

If $V(y)$ has an isolated minimum, then minimizing this energy leads, on the face of it, to a mathematical contradiction. Indeed, minimizing the potential energy leads us to a fixed value of y, while minimizing the kinetic energy leads us to a non-zero velocity.

That conundrum is connected to other peculiar aspects of the dynamics, which emerge from the equation of motion

$$(\dot{y}^2 - 1)\ddot{y} = -V'(y). \tag{3}$$

We see that the effective mass, $\dot{y}^2 - 1$, can vanish and change sign. Negative effective mass is unusual, though perhaps not problematic, but in this context (i.e., in the framework of Newtonian mechanics) vanishing effective mass signals that the evolution equation becomes either trivial or ill-defined. For those reasons, one might be inclined to think that the behavior implied by Eqn. (1) is inherently pathological, and physically unrealizable.

Here we will demonstrate that, to the contrary, Eqn. (1) arises as the effective description of a realistic physical system in an appropriate limit. The realization implies a specific regularization of the singular behavior. As the limit is approached, Eqn. (1) governs the behavior of the system accurately except during brief intervals. Thus, specifically, let us suppose that $V(y) = \frac{1}{2}y^2$ and that we choose the energy near its absolute minimum $-\frac{1}{4}$. Then, as long as Eqn. (2) applies, the system must have velocity near $\dot{y} = 1$ (or -1), yet stay close to $y = 0$. It can do this most of the time, if during very brief intervals the regulator comes into play, and allows a quick transit between small positive and small negative values of y. That is what we find to occur.

Thus, we discover that close to its energy minimum, solutions of the minimal time crystal Lagrangian, appropriately regulated, feature characteristic low-amplitude, high-frequency oscillations. The system behaves, in other words, as a tunable, non-dissipative relaxation oscillator, exhibiting temporal microstructure. Although here we will only discuss in detail a very specific example, involving a charged particle in special magnetic and electric fields, we anticipate that this kind of behavior can emerge, through the same mathematical mechanism, in many other contexts. In our concluding discussion we will elaborate on that expectation, and also suggest the possible utility of our regulator for describing spatial microstructure.

Model: Planar charge in external fields Consider the Lagrangian

$$L \;=\; \frac{\mu}{2}\dot{x}^2 + f(x)\dot{y} - g(x) - V(y). \tag{4}$$

This corresponds to a planar charged particle subjected to the magnetic field $B_z = f'(x)$ and the electric potential $g(x) + V(y)$. (For ease of exposition, we have taken the mass parameter in the y direction to vanish.). We have the equations of motion

$$\mu\ddot{x} = f'(x)\dot{y} - g'(x) \tag{5}$$

$$\dot{x}f'(x) = -V'(y). \tag{6}$$

Now let us consider the idealization $\mu \to 0$, which can be appropriate for strong magnetic fields. Then we have, from Eqn. (5), formally

$$\dot{y} = g'/f'. \tag{7}$$

Choosing $f(x) = \frac{1}{3}x^3 - x$, $g(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$, this becomes simply $\dot{y} = x$. Replacing x by $\dot{y}$ in the remaining equation of motion Eqn. (6) then reproduces Eqn. (3).

Alternatively, we can use Eqn. (5) to eliminate x from Eqn. (4) (with $\mu = 0$) to arrive at Eqn. (1) directly. (This demonstrates, *inter alia*, that our neglect of the mass parameter in the y direction is inessential in the time crystal regime, since Eqn. (1) already contains such a term. However, including this term expands the phase space, so that the time crystal effective theory corresponds only to a certain range of choices in the initial conditions.)

The sensitive point in this derivation is that in "deducing" Eqn. (7) we will, when $f' = 0$, have divided by zero. Physically, this is where the magnetic field vanishes, so that we can't neglect the mass μ, even if it parametrically small. Conversely, including a small positive μ acts as a regulator for the dynamical system defined by Eqn. (1). In this way, we have realized the minimal classical time crystal, including a potential, with a well-defined regulator, as a reduced description (effective theory) of a reasonably simple, physically realistic dynamical system.

Sisyphus Dynamics: Microstructure and Ratcheting Now let us explore the characteristic behavior of our regulated system more closely. Our system

$$\mu\ddot{x} = (x^2 - 1)(\dot{y} - x) \tag{8}$$

$$\dot{x}(x^2 - 1) = -y \tag{9}$$

is readily amenable to numerical study, which proves very revealing.

Before we display that characteristic behavior graphically, some preliminary comments are in order.

The initial value problem is well-posed with the specification of x and $\dot{x}$ at some initial time. In this formulation, y is a particularly interesting dependent variable, because we expect that it should reflect the time crystal dynamics directly.

The energy function is

$$E = \frac{\mu}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}y^2. \tag{10}$$

It is minimized by $x = \pm 1$ and $y = 0$, independent of time. At the minima, the energy is $E = -\frac{1}{4}$.

The characteristic "time crystal" temporal microstructure arises when the energy approaches, but is not equal to, that minimum value. Figure 1, the result of a numerical calculation using *Mathematica*, displays that behavior graphically. Note that the contribution $\frac{1}{2}\mu\dot{x}^2$ of the regulator to the energy is very small.

As one considers solutions whose energy approaches the minimum, the frequency of the oscillations in y increases while their amplitude decreases. The approach to the limit is, qualitatively, highly nonuniform. Quantitatively: though it is continuous in the norm $\mathrm{Max}\,|y|$, it is not continuous (for instance) in the norm $\mathrm{Max}\,|y| + \mathrm{Max}\,|\dot{y}|$, nor in simple Sobolev norms.

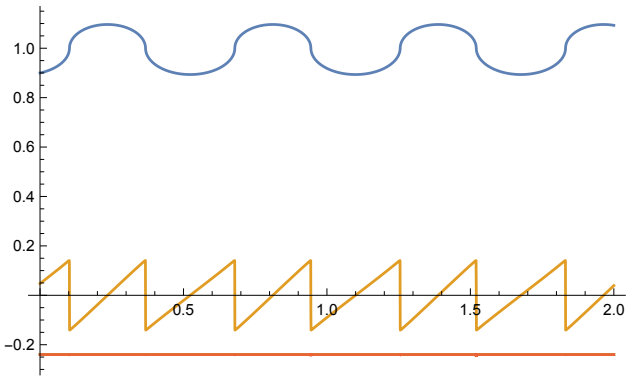


FIG. 1. Numerical solution of Eqns. (8,9) with $\mu = 10^{-5}$, $x(0) = 0.9$, $\dot{x}(0) = 0.25$. The upper (blue) curve represents $x(t)$; the middle (gold) curve represents $y(t)$; and the lower (red) curve represents E. The behavior of $y(t)$ exhibits the characteristic temporal microstructure discussed in the text, with near-constant velocity often, but briefly, interrupted by small, sudden jumps.

More generally: Within the effective theory, a significant energy barrier separates the positive velocity from the negative velocity region in velocity space. Thus, when the energy is too small to bridge the gap, the velocity will retain a constant sign. But that leads, as before, to trouble with the potential energy. So we might expect, in this more general situation, that the velocity is almost always positive (or almost always negative), interrupted by brief intervals when the effective theory breaks down, and the position gets reset. That behavior is indeed evident in the numerical simulations, as exemplified in Figure 2. Prominent in Figure 2, but also subtly present in Figure 1, is a diphasic structure in $y(t)$: the positive velocity evolves with two distinct patterns, depending upon whether x is greater than or less than unity.

Quantization: Suppression of small-amplitude, high-frequency oscillations, in Planck's theory of black-body radiation, was the first mission of quantum theory. Thus, it is appropriate to explore the quantum version of our model, to see how quantization reflects and modifies the (classical) temporal microstructure.

The most salient issue is the spectrum of bound states localized near the potential minimum. A first, heuristic step is to consider the semiclassical Bohr-Sommerfeld condition. It proves convenient to write this in its phase-space form, according to which the area A_n associated

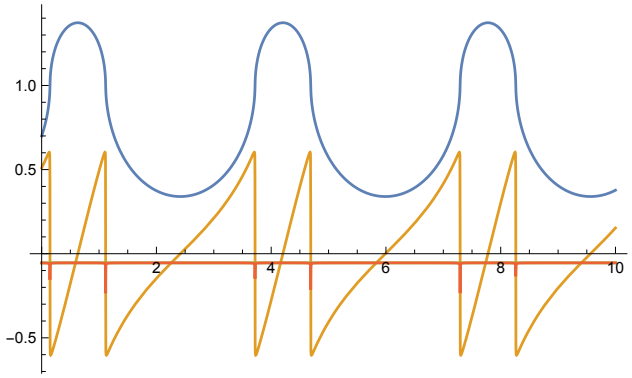


FIG. 2. Numerical solution of Eqns. (8,9) with $\mu = 10^{-3}$, $x(0) = 0.7$, $\dot{x}(0) = 1.0$. The color scheme is the same as in Figure 1. The behavior of $y(t)$ exhibits the characteristic ratcheting discussed in the text, with intervals of positive velocity interrupted by sudden jumps.

with energies less than the n^{th} eigen-energy E_n is approximately $2\pi\hbar n$, for n not too small. We have evaluated this condition in both the original time crystal (y) picture and the regulated (x) picture for small μ, assuming energy close to the classical minimum $-\frac{1}{4}$, with the concordant result

$$E_n + \frac{1}{4} \;=\; \left(\frac{3\pi\hbar n}{4\sqrt{2}}\right)^{\frac{2}{3}}. \tag{11}$$

Similar reasoning can be used to show that large, positive energy levels scale with a different power of n: $E_n \sim n^{4/5} + \mathcal{O}(1)$ as $n \to \infty$.

For rigorous quantization [16], we must pass to the regulated theory, and to a Hamiltonian formulation. For that purpose it is convenient to add a total derivative to Eqn. (4), so that $f(x)\dot{y} \to -yf'(x)\dot{x}$, and express everything in terms of x. Thus we find

$$H \;=\; \frac{p^2}{2(\mu + (1-x^2)^2)} + \frac{1}{4}x^4 - \frac{1}{2}x^2. \tag{12}$$

Now in passing to the quantum theory we meet an ordering ambiguity, since p and x do not commute. Pending deeper investigation we will adopt the ordering

$$H \;=\; -\frac{\hbar^2}{2}\,\rho^{-\frac{1}{4}}\,\frac{\partial}{\partial x}\,\rho^{-\frac{1}{2}}\,\frac{\partial}{\partial x}\,\rho^{-\frac{1}{4}} + \frac{1}{4}x^4 - \frac{1}{2}x^2 \tag{13}$$

$$\rho \;\equiv\; \mu + (1-x^2)^2 \tag{14}$$

as a simple prescription which leads to a Hermitian Hamiltonian. As a formal matter, we can vary the numerical value of $\hbar$, to reflect the relationships between other dimensional parameters we might have included (but did not) in Eqn. (1) and its descendants. Small values of $\hbar$ will emphasize the potential terms over the kinetic (gradient) terms, and thus de-emphasize the importance of the commutation relations, giving us the semiclassical limit. Large values of $\hbar$, conversely, take us into the deep quantum regime.

It is entirely practical to solve for the eigenvalues and eigenfunctions of Eqn. (13) numerically. (Indeed, the *Mathematica* command "NDEigensystem" makes it easy.) An important qualitative result that emerges is that the spectrum remains stable as $\mu \to 0$. Thus quantum mechanics itself regularizes the time crystal singularity, as anticipated.

For a more quantitative test, we can compare the low-lying eigenvalues of Eqn. (13) with the semiclassical result Eqn. (11) which, we recall, was derived directly from the time crystal Lagrangian. Figure 3 displays a representative result.

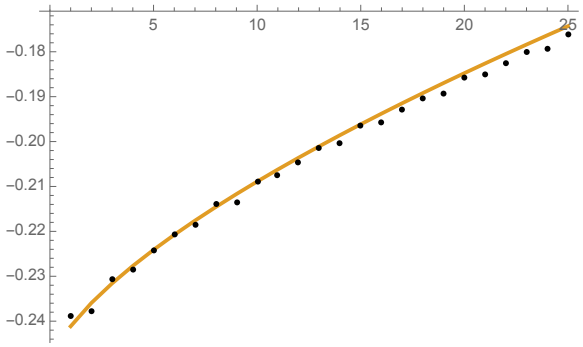


FIG. 3. Comparison of the parameter-free semiclassical predictions with numerically calculated values of the first twenty-five eigenvalues of Eqns. (13,14) with $\hbar = .0005$, $\mu = 10^{-7}$. The blue dots are the numerical calculations, the gold dots are the semiclassical predictions.

Discussion

1. The new variable x which is introduced in our regulation Eqn. (4) of the time crystal Lagrangian enters through a unique term involving $\dot{y}$, which is linear in $\dot{y}$. Thus it is a function of the momentum conjugate to y. In particular, it does not introduce new degrees of freedom. This should be contrasted with the (superficially) more straightforward approach of regulating singular behavior by adding higher derivatives [17, 18], i.e.

$$\Delta L \;=\; \epsilon\ddot{y}^2. \tag{15}$$

While this procedure manifestly overrides difficulties associated with vanishing coefficients in the highest-order terms in the equations of motion, it brings in other difficulties. Besides implicitly introducing new degrees of freedom, it also introduces instabilities [19].

2. We can introduce dissipation by adding a friction term $-\gamma\dot{x}$ on the right hand side of Eqn. (5). With this addition, the system generically evolves toward the energy minimum.

3. We can also encounter a form of Sisyphus dynamics in the purely spatial domain. Indeed, consider a time-independent system governed by the Hamiltonian (= − Lagrangian) density

$$H = \frac{1}{12}u_x^4 + \frac{b}{6}u_x^3 + \frac{c}{2}u_x^2 + \frac{1}{2}u^2. \tag{16}$$

We can regulate it using the same device as we used above for time crystal dynamics.

Varying, we find the stress equation

$$\partial_x T = 0 \tag{17}$$

$$T \equiv u_x^2(\frac{1}{4}u_x^2 + \frac{b}{3}u_x + \frac{c}{2}) - \frac{1}{2}u^2 \tag{18}$$

For a given solution, let $\langle T\rangle$ be the constant value of T, and let us suppose that $u^2_{\max}$ is the maximum value of u^2. (Since u^2 is energetically costly, the most interesting solutions are bounded in u^2.) Then $f \equiv u_x^2(\frac{1}{4}u_x^2 + \frac{b}{3}u_x + \frac{c}{2})$ satisfies

$$\langle T\rangle \leq f \leq \langle T\rangle + \frac{1}{2}u^2_{\max} \tag{19}$$

Now f, regarded as a function of u_x, defines the product of two parabolas. See Figure 4. As is evident from that Figure, for some choices of the parameters $b, c, \langle T\rangle, u^2_{\max}$, the values of u_x consistent with Eqn. (19) will be confined to one or two small, positive intervals, leading unambiguously to Sisyphus dynamics. For other values of b, c the allowed intervals may support both positive and negative values of u_x. See Figure 5. This opens up the possibility, commonly adopted in the micro-materials literature [17, 18], to keep u small by switching between positive and negative values of u_x, with appropriate joining prescriptions, e.g. using an ϵu^2_{xx} regulator. Our regulator suggests that Sisyphus dynamics, in effect allowing jumps in u rather than u_x, is a viable alternative here too.

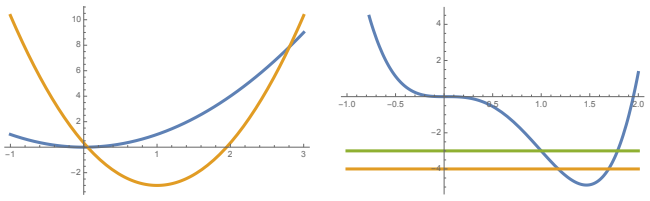


FIG. 4. On the left, two parabolas, one being simply a simple quadratic. On the right, in blue, their product, featuring a high non-degenerate positive minimum. When the value of the conserved stress lies between the green and orange straight lines, only positive values of the abscissa appear between them. As explained in the text, this leads to Sisyphus behavior.

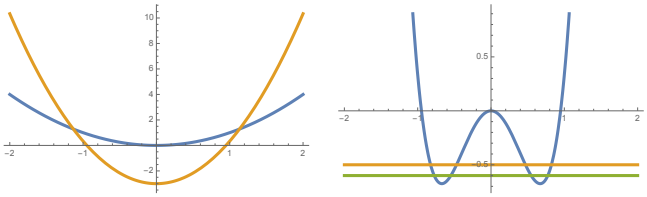


FIG. 5. Similar to Figure 4, but now both positive and negative values of the abscissa arise between the lines.

4. The Lagrangian $L = x(\dot{y}^2 - a) - \frac{1}{2}x^2 - V(y)$ represents a kind of self-consistent effective mass for y, together with an effective potential. Formally eliminating x now leads to the reduced Lagrangian $\frac{1}{2}(\dot{y}^2 - a)^2 - V(y)$, similar to what we had above. This illustrates that the emergence of "time crystal" effective Lagrangians is more general than the specific model which we analyzed in detail above.

5. We can consider a variation on Eqn. (4) using trigonometric functions. Taking $f = \frac{1}{3}\sin^3 x - \sin x$, $g = \frac{1}{4}\sin^4 x - \frac{1}{2}\sin^2 x$, and $\mu = 0$, we recover Eqn. (1). One can insert an appropriate regulator, and a parallel analysis then applies. Lagrangians of this sort describe periodic structures, and might also arise in the description of circuits including Josephson junctions. Those possibilities merit further investigation.

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