

# Entangled Histories

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February 8, 2015

## Abstract

We introduce quantum history states and their mathematical framework, thereby reinterpreting and extending the consistent histories approach to quantum theory. Through thought experiments, we demonstrate that our formalism allows us to analyze a quantum version of history in which we reconstruct the past by observations. In particular, we can pass from measurements to inferences about “what happened” in a way that is sensible and free of paradox. Our framework allows for a richer understanding of the temporal structure of quantum theory, and we construct history states that embody peculiar, non-classical correlations in time.

Many quantities of physical interest are more naturally expressed in terms of histories than in terms of “observables” in the traditional sense, i.e. operators in Hilbert space that act at a particular time. The accumulated phase  $\exp i \int_1^2 dt \vec{v} \cdot \vec{A}$  of a particle moving in an electromagnetic potential, or its accumulated proper time, are simple examples. We may ask: Having performed a measurement of this more general, history-dependent sort of observable, what have we learned? For conventional observables, the answer is that we learn our system is in a particular subspace of Hilbert space, that is the eigenspace corresponding to the observable’s measured value. Here we propose a general framework for formulating and interpreting history-dependent observables.

Over the last thirty years, the quantum theory of histories has been approached from several directions. In the 1980’s, Griffiths developed a mathematically precise formulation of the Copenhagen interpretation [1]. Griffiths was able to elucidate seemingly paradoxical experiments by enforcing a consistent interpretation of quantum evolution. Omnès, Gell-Mann, Hartle, Isham, and Linden, among others, enriched the mathematics and physics of Griffith’s theory of “consistent histories” [2]-[11]. In particular, Gell-Mann and Hartle focused on applying consistent histories to decoherence and quantum cosmology, while Isham and Linden’s work has uncovered deep mathematical structure at foundations of quantum mechanics [12].

Histories are of course an explicit element of Feynman’s path integral. In the early 1990’s, Farhi and Gutmann developed a generalized theory of the path integral, which clarifies what is meant by the path integral “trajectories” of a spin system, or any other system with non-classical features [13].

Here we will construct a formal structure that builds on these two lines of work, and illuminates the question posed in our first paragraph. We will also analyze several instructive examples, where we apply the formalism to analyze quasi-realistic thought experiments.

## 1 Mathematics of History States

### 1.1 History Space

We will work with a vector space that allows for access to the evolution of a system at multiple times. This vector space is called the history Hilbert space  $\mathcal{H}$ , and is defined by the tensor product from right to left of the admissible Hilbert spaces of our system at sequential times. Explicitly, for  $n$  times  $t_1 < \dots < t_n$ , we have

$$\mathcal{H} := \mathcal{H}_{t_n} \odot \dots \odot \mathcal{H}_{t_1} \tag{1}$$

where  $\mathcal{H}_{t_i}$  is the admissible Hilbert space at time  $t_i$ . Restricting the admissible Hilbert spaces  $\mathcal{H}_{t_1}$  and  $\mathcal{H}_{t_n}$  corresponds to pre- and post-selection respectively.

In this paper, we will be primarily concerned with history Hilbert spaces defined over a discrete set of times. It is possible to work with history Hilbert spaces over a continuum of times, but doing so requires the full apparatus of the Farhi-Gutmann path integral [13]. The history Hilbert space  $\mathcal{H}$  is also equipped with bridging operators  $T(t_j, t_i)$ , where  $T(t_j, t_i) : \mathcal{H}_{t_i} \rightarrow \mathcal{H}_{t_j}$ . These bridging operators encode unitary time evolution.

### 1.2 History States

We would like to define a mathematical object that encodes the evolution of our system through time – a notion of “quantum state” for history space – that supports an inner product and probability interpretation. One might at first think that an element of  $\mathcal{H}$  such as  $|\psi(t_n)\rangle \odot \dots \odot |\psi(t_1)\rangle$  would be the desired mathematical object, but a different concept appears more fruitful. For us, “history states” are elements of the linear space  $\text{Proj}(\mathcal{H})$  spanned by projectors from  $\mathcal{H} \rightarrow \mathcal{H}$ . Henceforth, we will call  $\text{Proj}(\mathcal{H})$  the “history state space.”

For example, if  $\mathcal{H}$  is the history space of a spin-1/2 particle at three times  $t_1 < t_2 < t_3$ , then an example history state is

$$[z^-] \odot [x^+] \odot [z^+] \tag{2}$$

where we use the notation  $[z^+] := |z^+\rangle\langle z^+|$ . The history state in Eqn. (2) can be considered as a quantum trajectory: the particle is spin up in the  $z$ -direction at time  $t_1$ , spin-up in

the  $x$ -direction at time  $t_2$ , and spin-down in the  $z$ -direction at time  $t_3$ . Since  $\text{Proj}(\mathcal{H})$  is a complex vector space, another example of a history state is

$$\alpha [z^-] \odot [x^+] \odot [z^+] + \beta [z^+] \odot [x^-] \odot [z^+] \quad (3)$$

for complex coefficients  $\alpha$  and  $\beta$ . The history state in Eqn. (3) is a superposition of the history states  $[z^-] \odot [x^+] \odot [z^+]$  and  $[z^+] \odot [x^-] \odot [z^+]$ . It can be interpreted, roughly, as meaning that the particle takes *both* quantum trajectories, but with different amplitudes.

Precise physical interpretation of history states like the one in Eqn. (3) requires more structure. For example, it is not obviously true (and usually is false) that one can measure the particle to take the trajectory  $[z^-] \odot [x^+] \odot [z^+]$  with probability proportional to  $|\alpha|^2$ , or the other trajectory with probability proportional to  $|\beta|^2$ . To discuss probabilities, generalizing the Born rule, we need an inner product. Furthermore, we have not yet defined which mathematical objects correspond to measurable quantities.

For a history space  $\mathcal{H}$  with  $n$  times  $t_1 < \dots < t_n$ , a general history state takes the form

$$|\Psi\rangle = \sum_i \alpha_i [a_i(t_n)] \odot \dots \odot [a_i(t_1)] \quad (4)$$

where each  $[a_i(t_j)]$  is a one-dimensional projector  $[a_i(t_j)] : \mathcal{H}_{t_j} \rightarrow \mathcal{H}_{t_j}$ , and  $\alpha_i \in \mathbb{C}$ . We have decorated the history state with a soft ket  $|\cdot\rangle$  which is suggestive of a wave function. In our theory, a history state is the natural generalization of a wave function, and has similar algebraic properties. Note that such sums of products of projectors will also accommodate products of hermitean operators more generally.

### 1.3 Inner Product

In defining a physically appropriate inner product between history states the  $K$  operator, defined by

$$K|\Psi\rangle = \sum_i \alpha_i K([a_i(t_n)] \odot \dots \odot [a_i(t_1)]) \quad (5)$$

$$= \sum_i \alpha_i [a_i(t_n)] T(t_n, t_{n-1}) [a_i(t_{n-1})] \odot \dots \odot [a_i(t_2)] T(t_2, t_1) [a_i(t_1)] \quad (6)$$

where  $T(t_j, t_i)$  is the bridging operator associated with the history space, plays a central role. Note that  $K$  maps a history state in  $\text{Proj}(\mathcal{H})$  to an operator which takes  $\mathcal{H}_{t_1} \rightarrow \mathcal{H}_{t_n}$ .

Using the  $K$  operator, we equip history states with the positive semi-definite inner product [1]

$$(\Phi|\Psi) := \text{Tr} \left[ (K|\Phi)\dagger K|\Psi\rangle \right] \quad (7)$$

This inner product induces a semi-norm on history states, and we call  $(\Psi|\Psi)$  the “weight” of  $|\Psi\rangle$ . It reflects the probability of  $|\Psi\rangle$  occurring. Note that the inner product is degenerate, in the sense that  $(\Psi|\Psi) = 0$  does not imply that  $|\Psi\rangle = 0$ .

We say that a history state  $|\Psi\rangle$  is normalized if  $\langle\Psi|\Psi\rangle = 1$ . If  $|\Psi\rangle$  has non-zero weight, then

$$|\bar{\Psi}\rangle = \frac{|\Psi\rangle}{\sqrt{\langle\Psi|\Psi\rangle}} \quad (8)$$

is normalized. We will use this “bar” notation throughout the rest of the paper.

At this point, an example may be welcome. If  $\mathcal{H}$  is the history space of a spin-1/2 particle at three times  $t_1 < t_2 < t_3$  equipped with a trivial bridging operator  $T = \mathbf{1}$ , then

$$K([z^-] \odot [x^+] \odot [z^+]) = \frac{1}{2} |z^-\rangle \langle z^+| \quad (9)$$

Let us interpret the factor of 1/2 on the right-hand side of the above equation. Since the bridging operator for the history space is the identity, any particle which is spin-up in the  $z$ -direction at time  $t_1$  will continue to be in that state at times  $t_2$  and  $t_3$ . However, the history state in Equation (9) is  $[z^-] \odot [x^+] \odot [z^+]$  which appears to subvert the unitary evolution imposed by the bridging operator. This subversion comes at a cost, which is the amplitude 1/2. We will later see that histories which do not follow unitary evolution have a suppressed probability of being measured, with the suppression factor proportional to the absolute square of the coefficient generated by the  $K$  operator. In the case of Equation (9), the “suppression” is a factor of  $|1/2|^2 = 1/4$ .

## 1.4 Families

We will be interested in subspaces that both admit an orthogonal basis (possibly including history states of zero norm), and contain the history state  $\mathbf{1}_{t_n} \odot \cdots \odot \mathbf{1}_{t_1}$ , which corresponds to a system being in a superposition of all possible states at each time. The orthogonal set of history states which spans such a subspace will be called a “family” of history states. More formally:

**Definition** We say  $\{|\bar{Y}^i\rangle\}$  is a family of history states if

- (1)  $\langle\bar{Y}^i|\bar{Y}^j\rangle = 0$  for  $i \neq j$  and  $\langle\bar{Y}^i|\bar{Y}^i\rangle = 0$  or 1, and
- (2)  $\sum_i c_i |\bar{Y}^i\rangle = \mathbf{1}_{t_n} \odot \cdots \odot \mathbf{1}_{t_1}$  for some complex  $c_i$ .

Requirement (1) is Griffiths’ “strong consistent histories condition” [1]-[3].

Contrary to earlier work, we see no reason to impose the requirement that history states, regarded as operators, commute. It is not essential that history states commute since they are not themselves observables. However, projectors of the form  $|\bar{Y}^i\rangle\langle\bar{Y}^i|$  are observables. By orthogonality,  $[\langle\bar{Y}^i|, |\bar{Y}^j\rangle]\langle\bar{Y}^j| = 0$  for all  $i, j$ , so commutativity of the corresponding observables is automatic. We also remark that a family of history states contains at most  $\dim(\mathcal{H}_{t_n}) \cdot \dim(\mathcal{H}_{t_1})$  history states with non-zero norm [14].

Now we will work through an example. Let us consider again the history space of a spin-1/2 particle at three times  $t_1 < t_2 < t_3$  equipped with a trivial bridging operator.

According to our definition,

$$\begin{aligned}
|\bar{Y}^1\rangle &= \sqrt{2}[z^+] \odot [x^+] \odot [z^+] + \sqrt{2}[z^-] \odot [x^-] \odot [z^+] \\
|\bar{Y}^2\rangle &= \sqrt{2}[z^-] \odot [x^+] \odot [z^+] + \sqrt{2}[z^+] \odot [x^-] \odot [z^+] \\
|\bar{Y}^3\rangle &= \sqrt{2}[z^+] \odot [x^+] \odot [z^-] + \sqrt{2}[z^-] \odot [x^-] \odot [z^-] \\
|\bar{Y}^4\rangle &= \sqrt{2}[z^-] \odot [x^+] \odot [z^-] + \sqrt{2}[z^+] \odot [x^-] \odot [z^-]
\end{aligned} \tag{10}$$

forms a family of history states. This family has the curious property that each history state is “entangled” in the sense that it is comprised of a linear combination of history states that is inseparable. We interpret each history state as an entangled quantum trajectory.

Let us explore some of the history states that live in  $\text{span}\{|\bar{Y}^1\rangle, |\bar{Y}^2\rangle, |\bar{Y}^3\rangle, |\bar{Y}^4\rangle\}$ . One such history state is  $|\Psi\rangle = [z^+] \odot [z^+] \odot [z^+]$ , which can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\bar{Y}^1\rangle + \frac{1}{\sqrt{2}}|\bar{Y}^2\rangle \tag{11}$$

Equation (11) implies that a state which at time  $t_1$  is spin up in the  $z$ -direction and evolves in time by the trivial bridging operator can be measured to be the history  $|\bar{Y}^1\rangle$  with probability  $|1/\sqrt{2}|^2 = 1/2$ , or the history  $|\bar{Y}^2\rangle$  with probability  $|1/\sqrt{2}|^2 = 1/2$ . Later we will outline how to make such a measurement.

Another interesting history state in  $\text{span}\{|\bar{Y}^1\rangle, |\bar{Y}^2\rangle, |\bar{Y}^3\rangle, |\bar{Y}^4\rangle\}$  is

$$|\Phi\rangle = \alpha[z^+] \odot [z^+] \odot [z^+] + \beta[z^-] \odot [z^-] \odot [z^-] \tag{12}$$

$$= \frac{\alpha}{\sqrt{2}}|\bar{Y}^1\rangle + \frac{\alpha}{\sqrt{2}}|\bar{Y}^2\rangle + \frac{\beta}{\sqrt{2}}|\bar{Y}^3\rangle + \frac{\beta}{\sqrt{2}}|\bar{Y}^4\rangle \tag{13}$$

which is normalized if  $|\alpha|^2 + |\beta|^2 = 1$ . The history state  $|\Phi\rangle$  is itself an entangled quantum trajectory. We will argue that it is possible to measure such objects.

## 1.5 Operators and Observables

Having provided several examples of history states, we will now briefly discuss operators on history states. In particular, we consider operators  $\hat{A}$  which are linear maps from history states to history states. In general, any operator of the form

$$\hat{A} : [\psi(t_n)] \odot \cdots \odot [\psi(t_1)] \mapsto \sum_i \alpha_i A_i^{t_n} [\psi(t_n)] (A_i^{t_n})^\dagger \odot \cdots \odot A_i^{t_1} [\psi(t_1)] (A_i^{t_1})^\dagger \tag{14}$$

where all  $A_i^{t_j}$  are linear operators, is a linear operator on history states.

As in standard quantum theory, not all operators correspond to observables. History state operators that correspond to observables are those which are both hermitean, and

whose eigenvectors can be extended to define a family. For example, given a family of history states  $\{|\bar{Y}^i\rangle\}$ , all history state operators of the form

$$\hat{B} = \sum_i b_i |\bar{Y}^i\rangle\langle\bar{Y}^i| \quad (15)$$

for  $b_i \in \mathbb{R}$  correspond to observables. A measurement of the normalized history state  $|\Psi\rangle$  by the observable  $\hat{B}$  gives the result  $b_i$  with probability  $|\langle\Psi|\bar{Y}^i\rangle|^2$  (or, for degenerate eigenvalues, the appropriate sum over such terms).

As another example, consider the history space of a spin-1/2 particle at two times  $t_1 < t_2$  equipped with a trivial bridging operator. We will consider the operator  $\sigma_y \odot \sigma_x$  which induces a linear map on history states by

$$\hat{C}|\Psi\rangle = \sum_i \alpha_i \sigma_y[\psi(t_2)]\sigma_y^\dagger \odot \sigma_x[\psi(t_1)]\sigma_x^\dagger \quad (16)$$

The  $\hat{C}$  operator corresponds to measuring the spin-1/2 particle at time  $t_1$  in the  $x$ -basis, and then measuring at time  $t_2$  in the  $y$ -basis. The eigenhistory states of  $\hat{C}$  form a family, namely  $\{\sqrt{2}[y^\pm] \odot [x^\pm]\}$  which are the history state “outputs” of the sequence of measurements. More generically, if our history space has a larger number of times, the linear map on history states induced by

$$\mathbf{1} \odot \cdots \odot \mathbf{1} \odot \sigma_y \odot \mathbf{1} \odot \cdots \odot \mathbf{1} \odot \sigma_x \odot \mathbf{1} \odot \cdots \odot \mathbf{1} \quad (17)$$

corresponds to measuring at some particular time in the  $x$ -basis followed by measuring at some later time in the  $y$ -basis.

We see that one way of thinking about measurements at one or more times is that they correspond to particular families of history states in which the history states at certain times are in eigenstates of observables. One can naturally consider measurements at one or more times that project onto multi-dimensional spaces, which allows one to control the complexity of a corresponding family of eigenhistory states.

For a more elaborate example of history state observables, we continue to consider a spin-1/2 particle at two times  $t_1 < t_2$  equipped with a trivial bridging operator. In addition to  $\sigma_y \odot \sigma_x$ , we will also consider  $\sigma_x \odot \sigma_z$  which induces a linear map on history states by

$$\hat{D}|\Psi\rangle = \sum_i \alpha_i \sigma_x[\psi(t_2)]\sigma_x^\dagger \odot \sigma_z[\psi(t_1)]\sigma_z^\dagger \quad (18)$$

The restrictions of  $\sigma_y \odot \sigma_x$  and  $\sigma_x \odot \sigma_z$  to either  $t_1$  or  $t_2$  do *not* commute. However,  $\sigma_x \odot \sigma_z$  and  $\sigma_z \odot \sigma_y$  themselves *do* commute, which reflects their non-trivial temporal structure. Thus,  $\sigma_x \odot \sigma_z$  and  $\sigma_z \odot \sigma_y$  have simultaneous eigenvectors, and correspondingly  $\hat{C}$  and  $\hat{D}$  have simultaneous eigenhistory states which in fact form a family. The simultaneous

eigenvectors of  $\sigma_y \odot \sigma_x$  and  $\sigma_x \odot \sigma_z$  are

$$|\Psi_1\rangle = -\frac{i}{2}|z^+\rangle \odot |z^+\rangle - \frac{1}{2}|z^+\rangle \odot |z^-\rangle - \frac{i}{2}|z^-\rangle \odot |z^+\rangle + \frac{1}{2}|z^-\rangle \odot |z^-\rangle \quad (19)$$

$$|\Psi_2\rangle = \frac{i}{2}|z^+\rangle \odot |z^+\rangle - \frac{1}{2}|z^+\rangle \odot |z^-\rangle + \frac{i}{2}|z^-\rangle \odot |z^+\rangle + \frac{1}{2}|z^-\rangle \odot |z^-\rangle \quad (20)$$

$$|\Psi_3\rangle = -\frac{i}{2}|z^+\rangle \odot |z^+\rangle + \frac{1}{2}|z^+\rangle \odot |z^-\rangle + \frac{i}{2}|z^-\rangle \odot |z^+\rangle + \frac{1}{2}|z^-\rangle \odot |z^-\rangle \quad (21)$$

$$|\Psi_4\rangle = \frac{i}{2}|z^+\rangle \odot |z^+\rangle + \frac{1}{2}|z^+\rangle \odot |z^-\rangle - \frac{i}{2}|z^-\rangle \odot |z^+\rangle + \frac{1}{2}|z^-\rangle \odot |z^-\rangle \quad (22)$$

and thus the simultaneous eigenhistory states of  $\hat{C}$  and  $\hat{D}$  are

$$|\Psi_1\rangle = \sqrt{2}|\Psi_1\rangle\langle\Psi_1| \quad (23)$$

$$|\Psi_2\rangle = \sqrt{2}|\Psi_2\rangle\langle\Psi_2| \quad (24)$$

$$|\Psi_3\rangle = \sqrt{2}|\Psi_3\rangle\langle\Psi_3| \quad (25)$$

$$|\Psi_4\rangle = \sqrt{2}|\Psi_4\rangle\langle\Psi_4| \quad (26)$$

Indeed, these history states are orthonormal and since  $\sum_i \frac{1}{\sqrt{2}}|\Psi_i\rangle = \mathbf{1}_{t_2} \odot \mathbf{1}_{t_1}$  we have a family.

Now that we have developed the necessary mathematical machinery, we will apply our framework to several model systems through quasi-realistic thought experiments. Our goal is to show that the mathematical machinery leads to physically sensible results.

## 2 Examples and Applications

### 2.1 Mach-Zehnder Interferometer

Consider the Mach-Zehnder interferometer in Figure 1:

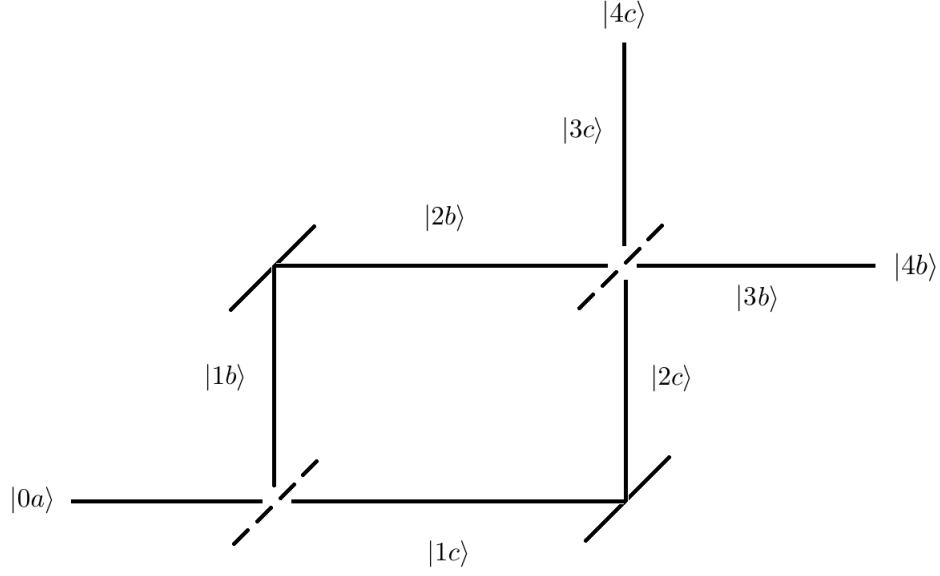


Figure 1. Diagram of a Mach-Zehnder interferometer.

The unitary time evolution of the system is

$$|0a\rangle \longrightarrow \frac{1}{\sqrt{2}} (|1b\rangle + |1c\rangle) \longrightarrow \frac{1}{\sqrt{2}} (|2b\rangle + |2c\rangle) \longrightarrow |3b\rangle \longrightarrow |4b\rangle \quad (27)$$

which displays interference. Note that the 50-50 beamsplitters act as the Hadamard matrix  $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  on the spatial modes. Equation (27) induces a history space of five times  $t_0 < t_1 < t_2 < t_3 < t_4$ , with a bridging operator which implements the unitary evolution of the system. Let us work with the family of history states

$$|\bar{\alpha}^1\rangle = 2 (|4c\rangle \odot \mathbf{1}_{t_3} \odot |2b\rangle \odot \mathbf{1}_{t_1} \odot |0a\rangle + |4b\rangle \odot \mathbf{1}_{t_3} \odot |2c\rangle \odot \mathbf{1}_{t_1} \odot |0a\rangle) \quad (28)$$

$$|\bar{\alpha}^2\rangle = 2 (|4c\rangle \odot \mathbf{1}_{t_3} \odot |2c\rangle \odot \mathbf{1}_{t_1} \odot |0a\rangle + |4b\rangle \odot \mathbf{1}_{t_3} \odot |2b\rangle \odot \mathbf{1}_{t_1} \odot |0a\rangle) \quad (29)$$

where  $|0a\rangle = \mathbf{1}_{t_0}$  since  $\mathcal{H}_{t_0} = \text{span}\{|0a\rangle\}$ . Notice that each history state is an entangled quantum trajectory. We have

$$\frac{1}{\sqrt{2}} |\bar{\alpha}^1\rangle + \frac{1}{\sqrt{2}} |\bar{\alpha}^2\rangle = \mathbf{1}_{t_4} \odot \mathbf{1}_{t_3} \odot \mathbf{1}_{t_2} \odot \mathbf{1}_{t_1} \odot \mathbf{1}_{t_0} \quad (30)$$

where the right-hand side corresponds to the history state in which the particle evolves unitarily, because we are not imposing in which state the particle should be at any time.



Equation (30) implies that we can measure a particle traveling through the Mach-Zehnder interferometer to be in the history state  $|\bar{\alpha}^1\rangle$  with probability  $|1/\sqrt{2}|^2 = 1/2$ , or  $|\bar{\alpha}^2\rangle$  with probability  $|1/\sqrt{2}|^2 = 1/2$ . We will now show how to make such a measurement.

To measure the unitary evolution of the Mach-Zehnder interferometer with respect to the  $|\bar{\alpha}^1\rangle, |\bar{\alpha}^2\rangle$  family, we couple the system at time  $t_0$  to an auxiliary qubit which lives in the space  $\text{span}\{|0\rangle, |1\rangle\}$ . We then evolve the original system in time while the auxiliary qubit goes along for the ride, and when appropriate, apply a CNOT gate (treating the auxiliary qubit as the target qubit) so that we can “mark” histories. The key point is that we are not allowed to meddle with the unitary evolution of the original system by imposing additional orthogonality relations via the auxiliary qubit. For example, since at time  $t_1$  the unitary evolution of the original system gives  $\frac{1}{\sqrt{2}}(|1b\rangle + |1c\rangle)$ , it is admissible for the auxiliary qubit to interact with the system by

$$\frac{1}{\sqrt{2}}(|1b\rangle + |1c\rangle) \otimes |0\rangle \quad \text{or} \quad \frac{1}{\sqrt{2}}(|1b\rangle \otimes |0\rangle + |1c\rangle \otimes |1\rangle) \quad (31)$$

but not

$$\frac{1}{2}|1b\rangle \otimes |0\rangle + \left(\frac{1}{2}|1b\rangle + \frac{1}{\sqrt{2}}|1c\rangle\right) \otimes |1\rangle \quad (32)$$

since the latter decoheres the system by imposing additional orthogonality.

The desired evolution of the combined Mach-Zehnder-qubit system is as follows:

$$|0a\rangle \otimes |0\rangle \xrightarrow{T(t_1, t_0) \otimes \mathbf{1}} \frac{1}{\sqrt{2}}(|1b\rangle \otimes |0\rangle + |1c\rangle \otimes |0\rangle) \quad (33)$$

$$\xrightarrow{T(t_2, t_1) \otimes \mathbf{1}, U_1} \frac{1}{\sqrt{2}}(|2b\rangle \otimes |0\rangle + |2c\rangle \otimes |1\rangle) \quad (34)$$

$$\xrightarrow{T(t_3, t_2) \otimes \mathbf{1}} \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}(|3b\rangle + |3c\rangle) \otimes |0\rangle + \frac{1}{\sqrt{2}}(|3b\rangle - |3c\rangle) \otimes |1\rangle \right) \quad (35)$$

$$\xrightarrow{T(t_4, t_3) \otimes \mathbf{1}, U_2} \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}(|4b\rangle - |4c\rangle) \otimes |0\rangle + \frac{1}{\sqrt{2}}(|4b\rangle + |4c\rangle) \otimes |1\rangle \right) \quad (36)$$

where we have

$$U_1 = |2b\rangle\langle 2b| \otimes |0\rangle\langle 0| + |2b\rangle\langle 2b| \otimes |1\rangle\langle 1| + |2c\rangle\langle 2c| \otimes |0\rangle\langle 1| + |2c\rangle\langle 2c| \otimes |1\rangle\langle 0| \quad (37)$$

$$U_2 = |4b\rangle\langle 4b| \otimes |0\rangle\langle 0| + |4b\rangle\langle 4b| \otimes |1\rangle\langle 1| + |4c\rangle\langle 4c| \otimes |0\rangle\langle 1| + |4c\rangle\langle 4c| \otimes |1\rangle\langle 0| \quad (38)$$

In this case, if we measure the auxiliary qubit at the final time  $t_4$  and detect  $|0\rangle$ , then the system has been in the history state  $|\bar{\alpha}^1\rangle$ , whereas if we detect  $|1\rangle$ , the system has been in the history state  $|\bar{\alpha}^2\rangle$ . Note that the probability amplitude of measuring the system to be in either  $|\bar{\alpha}^1\rangle$  or  $|\bar{\alpha}^2\rangle$  is  $1/\sqrt{2}$  which is reflected in Equation (30). Furthermore, measuring the auxiliary qubit at time  $t_4$  collapses the history state of the system to either  $|\bar{\alpha}^1\rangle$  or  $|\bar{\alpha}^2\rangle$ , each with probability  $1/2$ .

Note that at the final time  $t_4$ , it is not necessary to measure the auxiliary qubit in the  $\{|0\rangle, |1\rangle\}$  basis. Instead, we could measure the qubit in any other basis, such as the

$$\left\{ |+\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}|1\rangle, \quad |-\rangle = \sqrt{\frac{2}{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle \right\} \quad (39)$$

basis. In the  $\{|+\rangle, |-\rangle\}$  basis, Equation (36) takes the form

$$\begin{aligned} & \left[ \left( \frac{1}{2\sqrt{3}} - \frac{i}{\sqrt{6}} \right) |4b\rangle - \left( \frac{1}{2\sqrt{3}} + \frac{i}{\sqrt{6}} \right) |4c\rangle \right] \otimes |+\rangle \\ & + \left[ \left( \frac{1}{\sqrt{6}} + \frac{i}{2\sqrt{3}} \right) |4b\rangle + \left( -\frac{1}{\sqrt{6}} + \frac{i}{2\sqrt{3}} \right) |4c\rangle \right] \otimes |-\rangle \end{aligned} \quad (40)$$

Measuring  $|+\rangle$  at time  $t_4$  corresponds to measuring the history state

$$\frac{1}{\sqrt{3}}|\bar{\alpha}^1\rangle + i\sqrt{\frac{2}{3}}|\bar{\alpha}^2\rangle \quad (41)$$

and likewise  $|-\rangle$  corresponds to history state

$$\sqrt{\frac{2}{3}}|\bar{\alpha}^1\rangle - \frac{i}{\sqrt{3}}|\bar{\alpha}^2\rangle \quad (42)$$

Note that Equations (41) and (42) together form a family of history states for the Mach-Zehnder system, which is a linear transformation of our original  $\{|\bar{\alpha}^1\rangle, |\bar{\alpha}^2\rangle\}$  family. If two families of history states are related by a linear transformation, we say that they are “compatible.”

It is possible to measure a system in differing compatible families sequentially. For example, say that we want to measure the Mach-Zehnder system with respect to the  $\{|\bar{\alpha}^1\rangle, |\bar{\alpha}^2\rangle\}$  family followed by the family described by Equations (41) and (42). To do this, we tensor two auxiliary qubits to the initial state of the system, and evolve the system and auxiliary qubits in the same manner as before. At time  $t_4$ , we end up with

$$\frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} (|4b\rangle - |4c\rangle) \otimes |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} (|4b\rangle + |4c\rangle) \otimes |1\rangle \otimes |1\rangle \right) \quad (43)$$

If we measure the first auxiliary qubit in the  $\{|0\rangle, |1\rangle\}$  basis, then the system collapses to either  $|\bar{\alpha}^1\rangle$  or  $|\bar{\alpha}^2\rangle$  with equal probability. If we then measure the second qubit in the basis from Equation (39), the effect is to measure either the collapsed history state  $|\bar{\alpha}^1\rangle$  or  $|\bar{\alpha}^2\rangle$  in the compatible family described by Equations (41) and (42).

## 2.2 Spin-1/2 Particle

Once again, we consider the history space of a spin-1/2 particle at three times  $t_1 < t_2 < t_3$  with a trivial bridging operator. We will also impose that the particle be spin up in the  $z$ -direction at time  $t_1$ . Taking inspiration from Equation (10), we see that

$$|\bar{\beta}^1\rangle = \sqrt{2} [z^+] \odot [x^+] \odot [z^+] + \sqrt{2} [z^-] \odot [x^-] \odot [z^+] \quad (44)$$

$$|\bar{\beta}^2\rangle = \sqrt{2} [z^+] \odot [x^-] \odot [z^+] + \sqrt{2} [z^-] \odot [x^+] \odot [z^+] \quad (45)$$

is a family of history states for the system. In order to measure the unitary evolution of the system with respect to the  $|\bar{\beta}^1\rangle, |\bar{\beta}^2\rangle$  family, we couple an auxiliary qubit to the system at time  $t_1$  and evolve the system as follows:

$$|z^+\rangle \otimes |0\rangle \xrightarrow{T(t_2, t_1) \otimes \mathbf{1}, U_3} \frac{1}{\sqrt{2}} (|x^+\rangle \otimes |0\rangle + |x^-\rangle \otimes |1\rangle) \quad (46)$$

$$\xrightarrow{T(t_3, t_2) \otimes \mathbf{1}, U_4} \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} (|z^+\rangle - |z^-\rangle) \otimes |0\rangle + \frac{1}{\sqrt{2}} (|z^+\rangle + |z^-\rangle) \otimes |1\rangle \right) \quad (47)$$

where we have

$$U_3 = |x^+\rangle\langle x^+| \otimes |0\rangle\langle 0| + |x^+\rangle\langle x^+| \otimes |1\rangle\langle 1| + |x^-\rangle\langle x^-| \otimes |0\rangle\langle 1| + |x^-\rangle\langle x^-| \otimes |1\rangle\langle 0| \quad (48)$$

$$U_4 = |z^+\rangle\langle z^+| \otimes |0\rangle\langle 0| + |z^+\rangle\langle z^+| \otimes |1\rangle\langle 1| + |z^-\rangle\langle z^-| \otimes |0\rangle\langle 1| + |z^-\rangle\langle z^-| \otimes |1\rangle\langle 0| \quad (49)$$

If at the final time we measure the auxiliary qubit to be in the state  $|0\rangle$ , then the system has been in the history state  $|\bar{\beta}^1\rangle$ , and likewise if we measure the auxiliary qubit to be in the state  $|1\rangle$ , then the system has been in the history state  $|\bar{\beta}^2\rangle$ .

## 2.3 Decaying Particle Examples

### 2.3.1 A Single Decaying Particle

Consider for simplicity a particle that will either decay at time  $t_1$  with amplitude  $\alpha$  or  $t_2$  with amplitude  $\beta$ . If the particle decays at time  $t_1$ , it will flip an auxiliary state coupled to the system from  $|0\rangle$  to  $|1\rangle$ . Similarly, if the particle decays at time  $t_2$ , it will flip the auxiliary state from  $|0\rangle$  to  $|2\rangle$ . We assume that the auxiliary state otherwise evolves trivially in time. The history state of the system is

$$\alpha |\text{decayed at } t_1\rangle \otimes |1\rangle + \beta |\text{decayed at } t_2\rangle \otimes |2\rangle \quad (50)$$

where  $|1\rangle = [1] \odot \dots \odot [1]$  and  $|2\rangle = [2] \odot \dots \odot [2]$ . But if we measure the auxiliary state after time  $t_2$  in the  $\frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$  basis and post-select on measuring  $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ , then tracing out the auxiliary state we are left with the history state

$$\alpha |\text{decayed at } t_1\rangle + \beta |\text{decayed at } t_2\rangle \quad (51)$$

which is an entangled quantum trajectory.

### 2.3.2 Two Decaying Particles

If we extend the model above to two separate decaying particles and one auxiliary state for each particle, then the history state of the system is

$$\begin{aligned} & \alpha^2 |\text{decayed at } t_1\rangle|\text{decayed at } t_1\rangle \otimes |1\rangle|1\rangle + \alpha\beta |\text{decayed at } t_1\rangle|\text{decayed at } t_2\rangle \otimes |1\rangle|2\rangle \\ & + \beta\alpha |\text{decayed at } t_2\rangle|\text{decayed at } t_1\rangle \otimes |2\rangle|1\rangle + \beta^2 |\text{decayed at } t_2\rangle|\text{decayed at } t_2\rangle \otimes |2\rangle|2\rangle \end{aligned} \quad (52)$$

Measuring the auxiliary states after time  $t_2$  in the  $\left\{ \frac{1}{\sqrt{2}}(|1\rangle|2\rangle + |2\rangle|1\rangle), \dots \right\}$  basis, post-selecting on measuring  $\frac{1}{\sqrt{2}}(|1\rangle|2\rangle + |2\rangle|1\rangle)$ , and then tracing out the auxiliary states, we are left with the entangled history state

$$\frac{1}{|\alpha|^4 + |\beta|^4} \left( \alpha^2 |\text{decayed at } t_1\rangle|\text{decayed at } t_2\rangle + \beta^2 |\text{decayed at } t_2\rangle|\text{decayed at } t_1\rangle \right) \quad (53)$$

### 2.4 General Observations

It is clear from the above examples that the “time-evolution” picture with the conventional “ $\longrightarrow$ ” notation does not make the interplay between history states obvious. Our theory of history states allows us to manipulate time correlations and time entanglement in a way that is totally non-transparent otherwise. We also note that the most generic type of CNOT operator has the form

$$U = \sum_i |i\rangle\langle i| \otimes U_i \quad (54)$$

where  $\{|i\rangle\}$  is an orthonormal basis for our system of interest at some particular time, and each  $U_i$  is a unitary operator that acts on auxiliary qubits. These types of CNOT operators allow us to “mark” and “unmark” histories. If we choose the operators carefully, so as not to tamper with the unitary evolution of our system of interest by imposing, through the auxiliary qubits, additional orthogonality relations, we can use this construction to render the entities we have *defined* as history observables to *be* observable, concretely.

## 3 An Example of Extreme History Entanglement

In this section we consider an extreme example of history entanglement involving two particles. We will utilize the history space of two spin-1/2 particles at three times

$t_1 < t_2 < t_3$ , equipped with a trivial bridging operator. Then the history states

$$|\bar{Z}^1\rangle = 2[z^+, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] + 2[z^+, z^-] \odot [x^+, \mathbf{1}] \odot [z^-, x^+] \quad (55)$$

$$|\bar{Z}^2\rangle = 2[z^+, z^+] \odot [x^-, \mathbf{1}] \odot [z^+, x^+] + 2[z^+, z^-] \odot [x^-, \mathbf{1}] \odot [z^-, x^+] \quad (56)$$

$$|\bar{Z}^3\rangle = 2[z^+, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^-] + 2[z^+, z^-] \odot [x^+, \mathbf{1}] \odot [z^-, x^-] \quad (57)$$

$$|\bar{Z}^4\rangle = 2[z^+, z^+] \odot [x^-, \mathbf{1}] \odot [z^+, x^-] + 2[z^+, z^-] \odot [x^-, \mathbf{1}] \odot [z^-, x^-] \quad (58)$$

$$|\bar{Z}^5\rangle = 2[z^+, z^+] \odot [x^+, \mathbf{1}] \odot [z^-, x^+] + 2[z^+, z^-] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] \quad (59)$$

$$|\bar{Z}^6\rangle = 2[z^+, z^+] \odot [x^-, \mathbf{1}] \odot [z^-, x^+] + 2[z^+, z^-] \odot [x^-, \mathbf{1}] \odot [z^+, x^+] \quad (60)$$

$$|\bar{Z}^7\rangle = 2[z^+, z^+] \odot [x^+, \mathbf{1}] \odot [z^-, x^-] + 2[z^+, z^-] \odot [x^+, \mathbf{1}] \odot [z^+, x^-] \quad (61)$$

$$|\bar{Z}^8\rangle = 2[z^+, z^+] \odot [x^-, \mathbf{1}] \odot [z^-, x^-] + 2[z^+, z^-] \odot [x^-, \mathbf{1}] \odot [z^+, x^-] \quad (62)$$

$$|\bar{Z}^9\rangle = 2[z^-, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] + 2[z^-, z^-] \odot [x^+, \mathbf{1}] \odot [z^-, x^+] \quad (63)$$

$$|\bar{Z}^{10}\rangle = 2[z^-, z^+] \odot [x^-, \mathbf{1}] \odot [z^+, x^+] + 2[z^-, z^-] \odot [x^-, \mathbf{1}] \odot [z^-, x^+] \quad (64)$$

$$|\bar{Z}^{11}\rangle = 2[z^-, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^-] + 2[z^-, z^-] \odot [x^+, \mathbf{1}] \odot [z^-, x^-] \quad (65)$$

$$|\bar{Z}^{12}\rangle = 2[z^-, z^+] \odot [x^-, \mathbf{1}] \odot [z^+, x^-] + 2[z^-, z^-] \odot [x^-, \mathbf{1}] \odot [z^-, x^-] \quad (66)$$

$$|\bar{Z}^{13}\rangle = 2[z^-, z^+] \odot [x^+, \mathbf{1}] \odot [z^-, x^+] + 2[z^-, z^-] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] \quad (67)$$

$$|\bar{Z}^{14}\rangle = 2[z^-, z^+] \odot [x^-, \mathbf{1}] \odot [z^-, x^+] + 2[z^-, z^-] \odot [x^-, \mathbf{1}] \odot [z^+, x^+] \quad (68)$$

$$|\bar{Z}^{15}\rangle = 2[z^-, z^+] \odot [x^+, \mathbf{1}] \odot [z^-, x^-] + 2[z^-, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] \quad (69)$$

$$|\bar{Z}^{16}\rangle = 2[z^-, z^+] \odot [x^-, \mathbf{1}] \odot [z^-, x^-] + 2[z^-, z^-] \odot [x^-, \mathbf{1}] \odot [z^+, x^-] \quad (70)$$

form a family.

Each history state is an entangled quantum trajectory, and the entanglement encodes novel physical behavior. Consider, specifically,

$$|\bar{Z}^1\rangle = 2[z^+, z^+] \odot [x^+, \mathbf{1}] \odot [z^+, x^+] + 2[z^+, z^-] \odot [x^+, \mathbf{1}] \odot [z^-, x^+]$$

This history state exhibits “time entanglement”: Measuring the first particle at time  $t_1$  does not determine the state of the second particle until time  $t_3$ . We have that the state of particle 1 at time  $t_1$  is the same as the state of particle 2 at time  $t_3$  – behavior in time similar to that which a Bell state exhibits in space. Using constructions similar to the ones in the previous section, it is possible to measure such a history state. Such extreme entanglement exemplifies the possibility of new structures emerging from the quantum theory of history states.

## 4 Relation to Operator Calculus

In the 1950s, Feynman introduced a generalized operator calculus, which enables useful algebraic manipulations in quantum mechanics, and ultimately quantum field theory [15]. In this brief section we will show that Feynman’s operator calculus finds a natural home in history space.

Let  $\mathcal{A}$  be the space of operators on some space  $\mathcal{X}$ , where  $\mathcal{A}$  includes the identity operator  $\mathbf{1}$ . We define an indexed operator of order  $n$  as an element of the space

$$\underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A} \otimes \mathcal{A}}_{n \text{ of these}} \quad (71)$$

If  $\alpha \in \mathcal{A}$ , we denote by  $\alpha_i$

$$\alpha_i := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \alpha \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \quad (72)$$

where the  $\alpha$  operator is in the  $i$ th spot from the right. Note that  $\alpha_i$  is an indexed operator of order  $n$ . Next, we define the map

$$K : \mathcal{A} \otimes \cdots \otimes \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \quad (73)$$

by

$$K : \gamma \otimes \cdots \otimes \beta \otimes \alpha \longmapsto \gamma \cdots \beta \alpha \quad (74)$$

which resembles the action of the  $K$  operator in the previous sections. This gives us

$$K(\alpha_i \beta_j) = \begin{cases} \beta \alpha & \text{if } i < j \\ \alpha \beta & \text{if } i = j \\ \alpha \beta & \text{if } i > j \end{cases} \quad (75)$$

which is the heart of Feynman’s operator calculus. We can interpret Feynman’s expressions

$$f(\alpha_i + \beta_j) \text{ “=” some combination of } \alpha' \text{’s and } \beta' \text{’s}$$

for some function  $f$  according to

$$f(\alpha_i + \beta_j) \text{ “=” } K f(\alpha_i + \beta_j) \quad (76)$$

## 5 Conclusion

The framework of history states elucidates the temporal structure of quantum theory and makes sense of an expanded class of observables which act on quantum trajectories. We have constructed examples of entangled history states with non-classical correlations in time which can be experimentally realized. It would be very interesting to connect these ideas to the mathematical theory of (classical) inference and causality, which has matured in recent years [16].

## Acknowledgements

Jordan Cotler was funded by the Undergraduate Research Opportunities Program at the Massachusetts Institute of Technology. Frank Wilczek's work is supported by the U.S. Department of Energy under grant Contract Number DE-SC00012567.

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