

The Physics of Axions

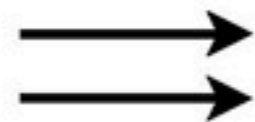
- **Why** the standard model is flawed, and we're motivated to tinker with it.
- **What** Peccei-Quinn symmetry and axions are.
- **Whether** they exist: their signatures in physics and cosmology.
- **How** these ideas interact with unification and supersymmetry.

Introductory Flowchart

QCD
↓ ↓
 $U_A(1)$ problem



Instantons,
topological
interactions

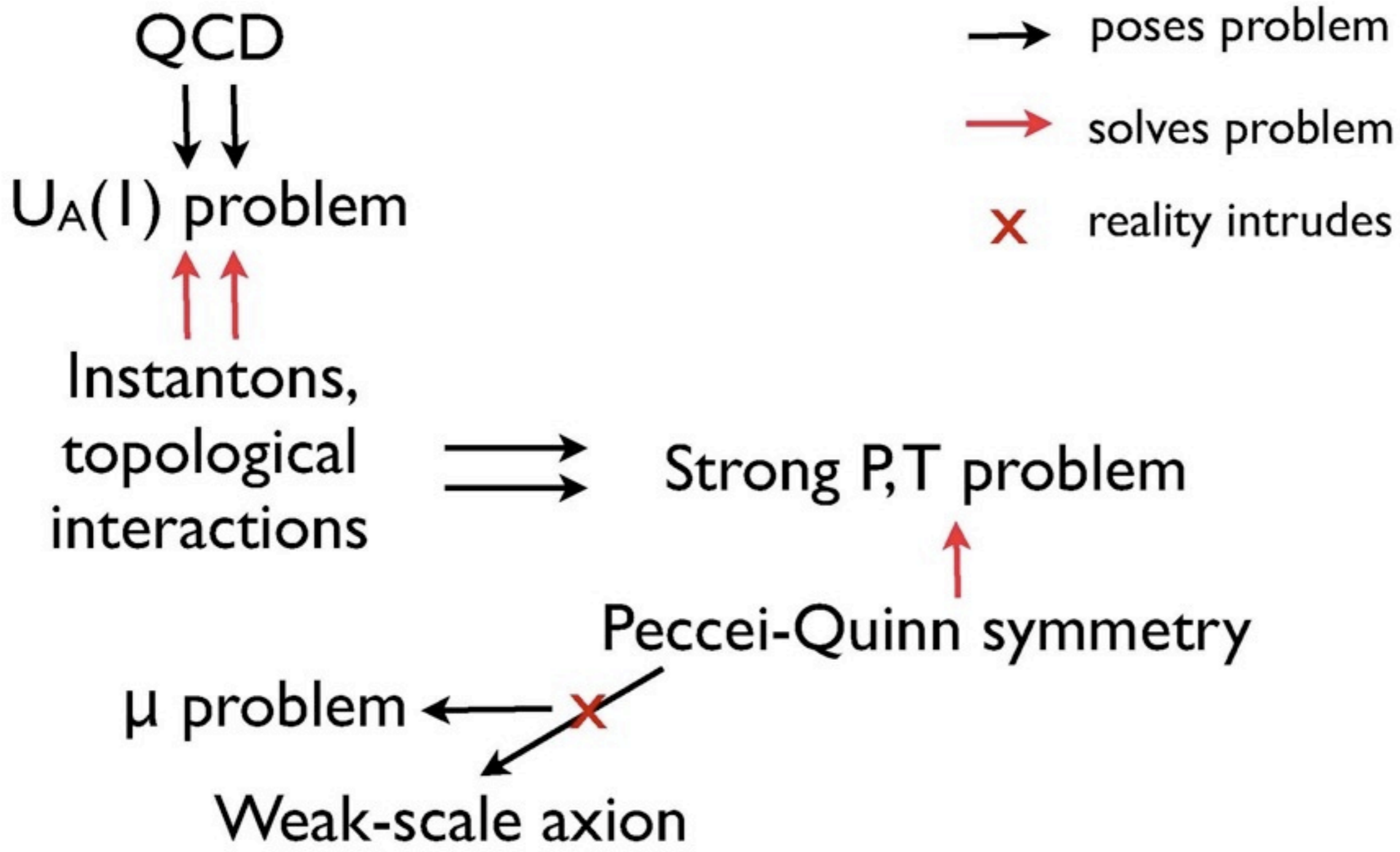


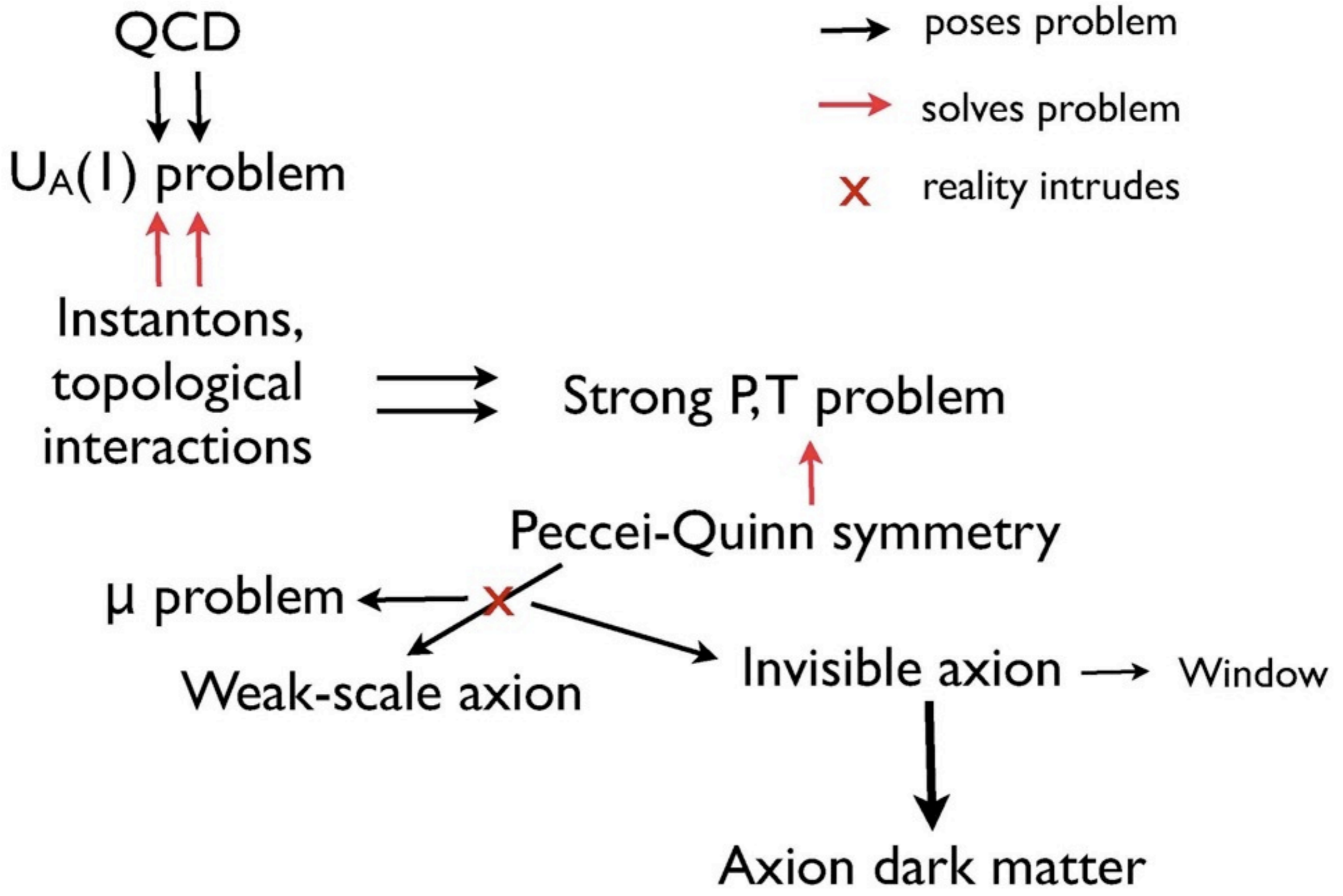
Strong P,T problem

Peccei-Quinn symmetry

- poses problem
- solves problem
- × reality intrudes

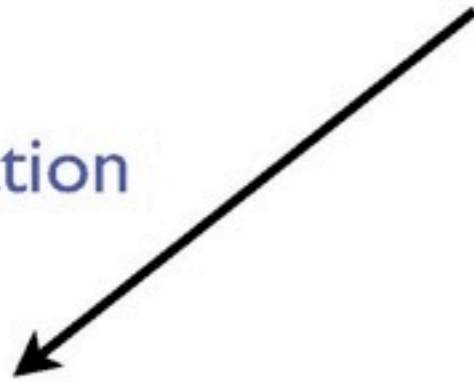






Axion dark matter

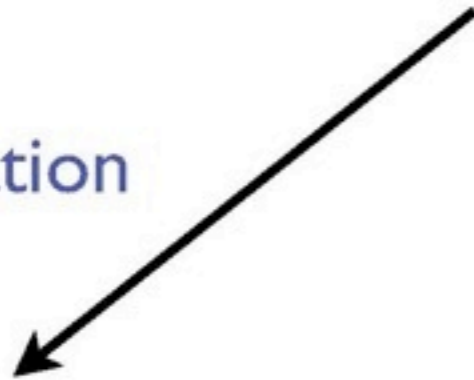
PQ < inflation



Antenna design

Axion dark matter

PQ < inflation



Antenna design

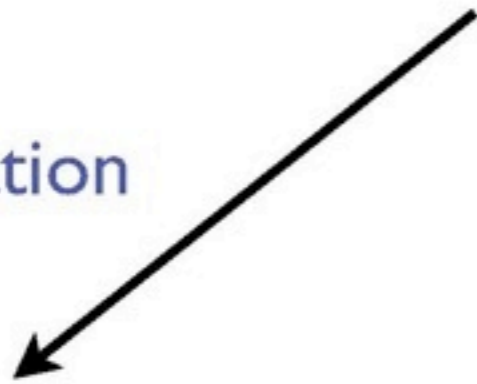
PQ > inflation



Selection

Axion dark matter

PQ < inflation

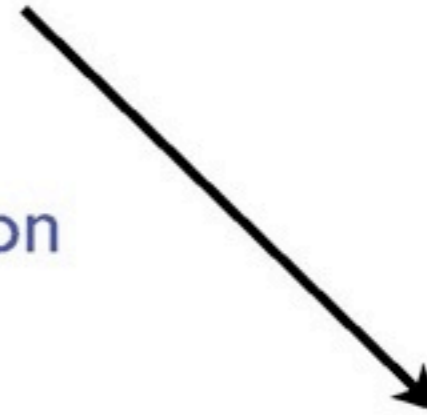


Antenna design

PQ > inflation



Selection



Isocurvature
Fluctuations

T-Symmetry in Fundamental Physics

Old and New Issues

Observed T (and CP) Violation

Exotic Spice From Flavor Physics

In practical experience the distinction between past and future is stark.

Nevertheless *microscopic* time-reversal symmetry (T) is a feature of the classic equations of physics - e.g. Newton, Maxwell, Schrodinger - and was still thought to be valid more than halfway through the 20th century.

The first crack came with the downfall of spatial reflection or parity symmetry (P) in 1956. It was soon realized that P violation in the weak interactions is in some sense “maximal”. This led to V-A theory and the concept of chirality, which are dominant features of modern fundamental physics.

However there was a fallback position: combined parity-charge conjugation (CP) appeared to be valid even in weak interactions.

Validity of CPT is guaranteed by very general considerations in local relativistic quantum field theory. Assuming this, CP is equivalent to T.

In 1964 Fitch, Cronin, Christensen and Turlay discovered a small, subtle violation of CP in K meson decays. For many years, this was the only setting in which CP violation was observed.

In 1973 Kobayashi and Maskawa proposed a specific mechanism for CP violation in the context of our Core theory (standard model).

According to KM's analysis, CP *conservation* is an “accidental” consequence of other symmetries in the Core theory with two families.

At that time only two families were known, so KM proposed there must be a third.

The third family was established not long after.

Modern studies in B-meson physics have vindicated KM's mechanism in great detail.

They reveal that CP violation is not parametrically small (but to see it one must involve all three families).

On the other hand *extremely* accurate searches for another T-violating effect, the possible existence of elementary electric dipole moments, have so far come up empty.

An elementary dipole moment corresponds to a term $\Delta H \propto \mathbf{S} \cdot \mathbf{E}$, which manifestly violates both T and P.

Experimentally, $|d_n| < 6 \times 10^{-26}$ e-cm. [See M. Pospelov and A. Ritz, hep-ph/0504231, for atoms and leptons too.]

The bounds are striking. In particular electric dipole moments are many orders of magnitude smaller than the “natural” size $e \cdot (\text{Compton wavelength})$

Explaining It All (Almost)

Operator Analysis in the Standard Model

The structure of the Core (= standard model) sheds much light on the patterns of T (and P) violation sketched above.

[analysis of canonical forms]

Thus we see how the deep structure of the Core appears to explain how to capture maximal P violation, why CP violation appears only in flavor-changing processes and is hidden at low energies, ...

... and in particular why T violating electric dipole moments are heavily suppressed.

Problem of Strong T Symmetry

(The Gap in the Argument)

Now we'll see how this pleasant package unravels!

There is an important difference between the symmetries of the classical Lagrangian of QCD and the symmetries of the quantum theory.

It solves a major problem within QCD, but poses a major puzzle for the Core as a whole.

I. Why we don't want
 $U_A(I)$

In the approximation $m_u = m_d = 0$, QCD appears (classically) to exhibit chiral flavor $U(2)_L \times U(2)_R$ symmetry, allowing unitary “isospin” transformations between both (u_L, d_L) and (u_R, d_R) , independently.

The diagonal $U(1)$ - a common phase for all four - corresponds to baryon number conservation.

A condensate of the type $\langle \bar{u}_L u_R \rangle = \langle \bar{d}_L d_R \rangle = v \neq 0$ develops. (Once this was an ingenious hypothesis; now it is a computed fact.)

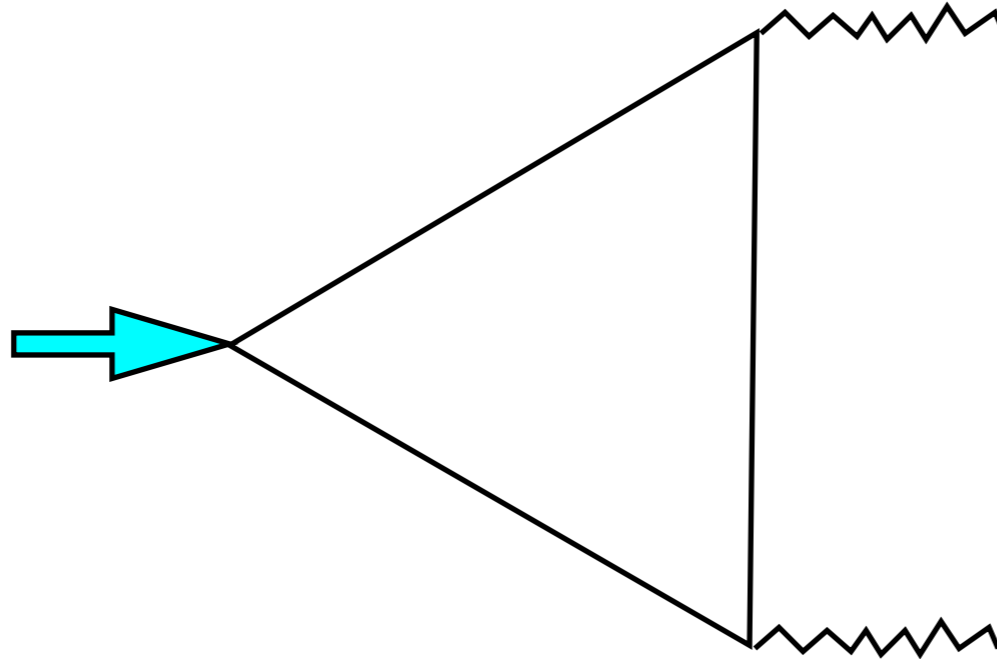
The spontaneous breaking of $SU(2)_L \times SU(2)_R \rightarrow SU(2)_{L+R}$ results in three Nambu-Goldstone bosons. This is the basis for a very successful theory of pions.

Axial baryon number symmetry, under which left handed and right handed quarks acquire opposite phases, is also spontaneously broken by this condensate. However, in this case there is no suitable candidate for the (approximate) Nambu-Goldstone boson.

Including effects of small quark masses, and strangeness, does not improve the situation.

2. Why we don't get
 $U_A(I)$

Careful computation of quantum behavior, which involves regulating the divergences due to highly virtual particles (ultraviolet divergences), reveals an *anomaly* in the axial baryon number current:



$$\partial_\mu(\bar{u}\gamma_5\gamma^\mu u + \bar{d}\gamma_5\gamma^\mu d) \equiv \partial_\mu j^{5\mu} \propto \text{Tr} \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta} G_{\gamma\delta}$$

First reaction: So what? There's a modified, conserved current:

$$\begin{aligned}\text{Tr } \epsilon^{\alpha\beta\gamma\delta} G_{\alpha\beta} G_{\gamma\delta} &= \partial_\alpha 4 \text{Tr } \epsilon^{\alpha\beta\gamma\delta} (A_\beta \partial_\gamma A_\delta + \frac{2}{3} A_\beta A_\gamma A_\delta) \equiv \partial_\alpha K^\alpha \\ \tilde{j}^{5\mu} &= j^{5\mu} + K^\mu\end{aligned}$$

Second thought: K^μ is gauge dependent; it might be singular.

By considering Euclidean functional integrals, 't Hooft demonstrated, in a semiclassical approximation, that it *is* singular.

Central point: You can have field configurations with finite weight (finite $\int \text{Tr } G_{\mu\nu} G^{\mu\nu}$) for which $\int K^\mu$ diverges. This can occur if $G_{\mu\nu} \rightarrow 0$, due to cancellations between $\partial_\mu A_\nu$ and $[A_\mu, A_\nu]$ that do not occur in K^μ .

$G_{\mu\nu} \rightarrow 0$ indicates a pure gauge configuration. This is on the verge of being trivial. However, topology saves the day. The “singularity” (reflected in K) is locally trivial, but may be globally nontrivial.

This begins a long and interesting mathematical story. Here I’ll go directly to the punch line:

Finite-action configurations can contribute to $\int \text{Tr} \varepsilon^{\alpha\beta\gamma\delta} \mathbf{G}_{\alpha\beta} \mathbf{G}_{\gamma\delta}$, and thus (according to the anomaly) spoil conservation of $j^{5\mu}$.

The contributions come in integer multiples of $16\pi^2$.

Thus the physical effect of

$$\Delta L_{\text{Euc.}} = i \theta (16\pi^2)^{-1} \int \text{Tr} \varepsilon^{\alpha\beta\gamma\delta} \mathbf{G}_{\alpha\beta} \mathbf{G}_{\gamma\delta}$$

is 2π periodic in θ .

Note: Under P or T, $\theta \rightarrow -\theta$!

What about the anomaly equation? How does $U_A(I)$ get violated, concretely?

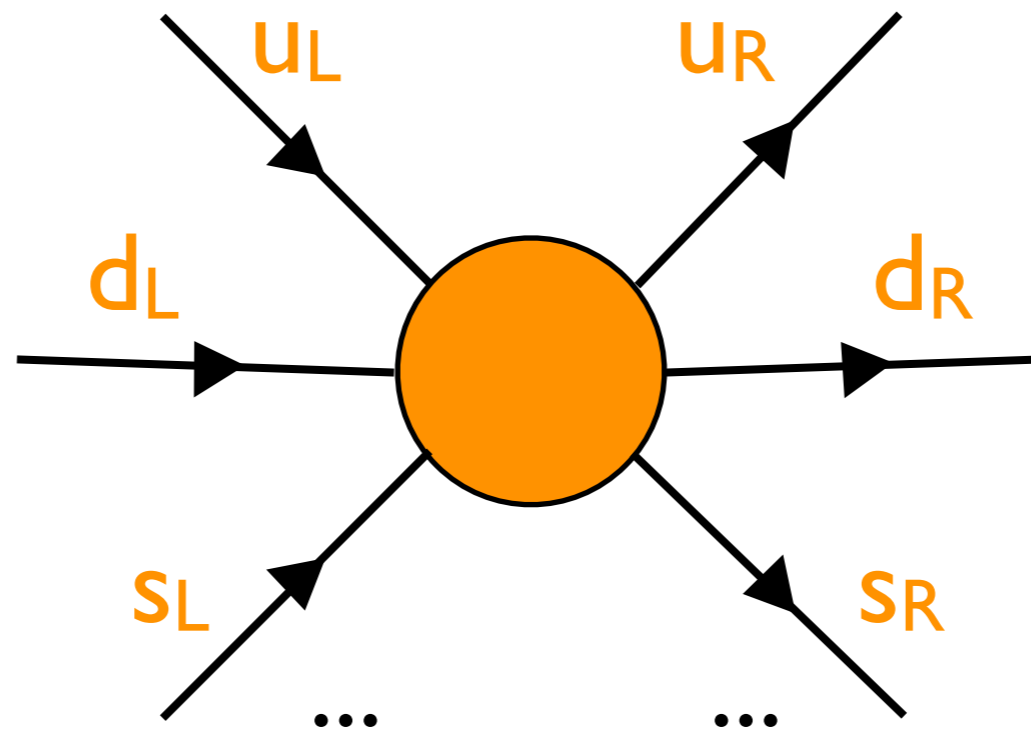
In topologically non-trivial backgrounds, there are (4d) “zero energy” solutions of the Dirac equation. These must be saturated, in order to get a non-zero answer (since the fermion integral yields a determinant).

The zero modes have one chirality in the imaginary-time past, another in the imaginary-time future.

3. How to picture all this

Summary: Topological quasi-singularities in the (4d, Euclidean) gauge field generate a new effective interaction in the quantum field theory of QCD, that is absent in the classical theory. It spoils the anomalous $U_A(1)$ symmetry.

A picture worth a thousand words: the 't Hooft vertex:



$$e^{i\theta}$$

The behavior suggested by the foregoing semiclassical analysis - non-conservation of $U_A(1)$; absence of an “extra” Nambu-Goldstone boson - has now been validated in fully non-perturbative, numerical calculations in QCD and related theories.

4. Why it's trouble

The bad news is that the failure of this symmetry means that we can't argue away the overall phase of the quark mass matrix, nor the $\text{Tr}\epsilon^{\alpha\beta\gamma\delta}\mathbf{G}_{\alpha\beta}\mathbf{G}_{\gamma\delta}$ term in the Lagrangian.

(We can shuffle from one to the other, as indicated by the 't Hooft vertex and the anomaly equation.)

These terms violate P and T, but do not change flavor! Thus they contribute directly, and *strongly*, to electric dipole moments.

Phenomenologically, one deduces $|\theta_{\text{eff.}}| < 10^{-10}$ or so.

Why?? (Note: Not required anthropically.)

Summary

As a working picture, the idea of CP violation solely due to a complex phase in the weak currents, is **remarkably** successful.

Its success raises deep theoretical questions, **especially:**

Why is the θ term of QCD is so small?

How Addressing Strong T-Symmetry Suggests Strange New Particles

Axions

Two Models of Quark Masses

Minimal but Unnatural; (Slightly) Non-Minimal but
Natural

Going one level deeper in the generation of quark masses, in the standard model and a slight variant:

(Minimal) standard model:

$$g_{jk} \phi^\alpha \bar{L}_\alpha^j U^k + h_{jk} \epsilon^{\alpha\beta} \phi_\beta^* \bar{L}_\alpha^j D^k + \text{h.c.}$$

The diagram shows the hypercharge assignments for the fields in the Lagrangian. For the first term, $g_{jk} \phi^\alpha \bar{L}_\alpha^j U^k$, the hypercharges are: ϕ^α has $-1/2$, \bar{L}_α^j has $-1/6$, and U^k has $2/3$. For the second term, $h_{jk} \epsilon^{\alpha\beta} \phi_\beta^* \bar{L}_\alpha^j D^k$, the hypercharges are: ϕ_β^* has $1/2$, \bar{L}_α^j has $-1/6$, and D^k has $-1/3$.

Variant:

$$g_{jk}\phi_1^\alpha \bar{L}_\alpha^j U^k + h_{jk}\epsilon^{\alpha\beta}\phi_{2\beta} \bar{L}_\alpha^j D^k + \text{h.c.}$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ -1/2 \quad | \quad | \\ \quad \quad -1/6 \quad | \\ \quad \quad \quad \quad 2/3 \end{array}$$

$$\begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ 1/2 \quad | \quad | \\ \quad \quad -1/6 \quad | \\ \quad \quad \quad \quad -1/3 \end{array}$$

The minimal standard model contains just one Higgs doublet.

The variant contains two doublets with opposite hypercharge.

The field content of the variant could support 2 additional couplings with $\varepsilon^{\alpha\beta}$. They could be forbidden by additional symmetries, however.

Minimal weak-scale supersymmetry includes just this sort of variant structure.

(Important methodological point: In either scheme, the observable CKM mixings are complicated functions of the basic g , h , and $\langle\Phi_1\rangle/\langle\Phi_2\rangle$.)

$$g_{jk}\phi^\alpha \bar{L}_\alpha^j U^k + h_{jk}\epsilon^{\alpha\beta}\phi_\beta^* \bar{L}_\alpha^j D^k + \text{h.c.}$$

In the minimal (one Higgs doublet) standard model, the overall phase of the quark mass matrix is a definite function of the parameters g and h , namely $\text{Arg det } g \text{ det } h$. The phase of Φ is irrelevant. Thus the smallness of strong P,T violation goes unexplained; it requires “fine tuning”.

$$g_{jk}\phi_1^\alpha \bar{L}_\alpha^j U^k + h_{jk}\epsilon^{\alpha\beta}\phi_{2\beta}\bar{L}_\alpha^j D^k + \text{h.c.}$$

In the variant model the phase of the quark mass matrix becomes a *dynamical variable*. It is $\text{Arg} \langle \Phi_1 \rangle \langle \Phi_2 \rangle \det g \det h$.

This opens a possibility to explain the smallness of strong P,T violation dynamically.

Axions I

In the Toy Model

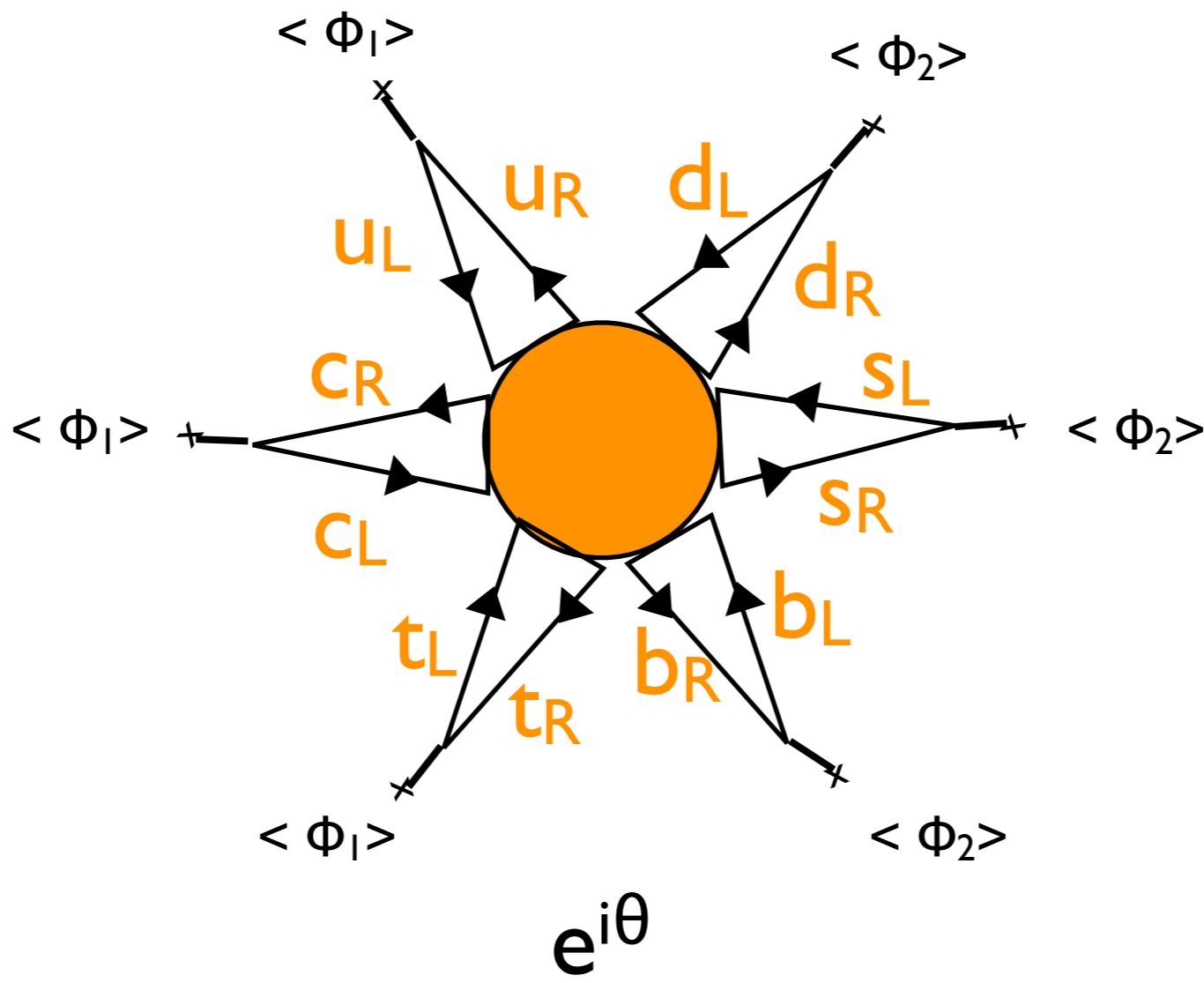
Promoting $\theta_{\text{eff.}}$ to a dynamical variable is not enough. We want to make sure that it settles down close to 0!

This requires, first, that the energy associated with the total phase of $\Phi_1\Phi_2$ should be determined primarily by $\theta_{\text{eff.}}$, or (roughly speaking) by the 't Hooft interaction.

To insure that, we impose Peccei-Quinn (PQ) symmetry: The classical Lagrangian should be invariant under $\Phi_1 \rightarrow e^{i\sigma} \Phi_1$, $\Phi_2 \rightarrow e^{i\sigma} \Phi_2$.

This forbids, in particular, terms proportional to $\Phi_1 \Phi_2$ or $(\Phi_1 \Phi_2)^2$, which might otherwise have appeared.

Now consider how the total phase of $\Phi_1\Phi_2$ does affect the vacuum energy:



$\theta_{\text{eff.}}$, the phase that multiplies the 't Hooft vertex in vacuum, is exactly what figures in the energy.

The energy is indeed minimized at the symmetry point $\theta_{\text{eff.}} = 0$.

The spontaneous breaking of a (global) symmetry is accompanied by a Nambu-Goldstone boson, with characteristic properties:


It is massless.

It couples gradiently to the symmetry current.

The strength of coupling is inversely proportional to the scale of symmetry breaking.

$$\phi \rightarrow e^{i\alpha} \phi \quad \langle \phi \rangle = F$$

$$\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi^* + \mathcal{L}_{\text{int.}} \approx F^2 \partial_\mu \alpha \partial^\mu \alpha + \partial_\mu \alpha \frac{\delta \mathcal{L}}{\delta \partial_\mu \alpha}$$


 Noether current

$$F\alpha \equiv b \quad \mathcal{L} \sim \partial_\mu b \partial^\mu b + \frac{1}{F} \partial_\mu b j^\mu$$

Here we get the same structure, augmented by *intrinsic* breaking through the anomalies (including the 't Hooft vertex).

We can get the basic idea simply by expanding, though a more refined analysis should take all forms of spontaneous symmetry breaking into account simultaneously (e.g. there is some a - π^0 mixing).

[minimal axion equations]

Axioms 2

In General

In modern high-energy theory we've become comfortable thinking big.

Specifically, in thinking about unification, we routinely contemplate mass scales well beyond the weak scale.

Could Peccei-Quinn symmetry be broken at a large scale?

Indeed, it is not difficult to make models with high-scale PQ breaking. Most simply, we introduce a standard model singlet complex scalar field ρ that both

transforms non-trivially under PQ symmetry, and

acquires a large vacuum expectation value F .

[generalized axion equations]

It is important that we don't have too much symmetry; in particular, we should break the "old" PQ symmetry, involving just Φ_1 and Φ_2 . To insure this, we allow interactions of the type $\rho \Phi_1 \Phi_2$.

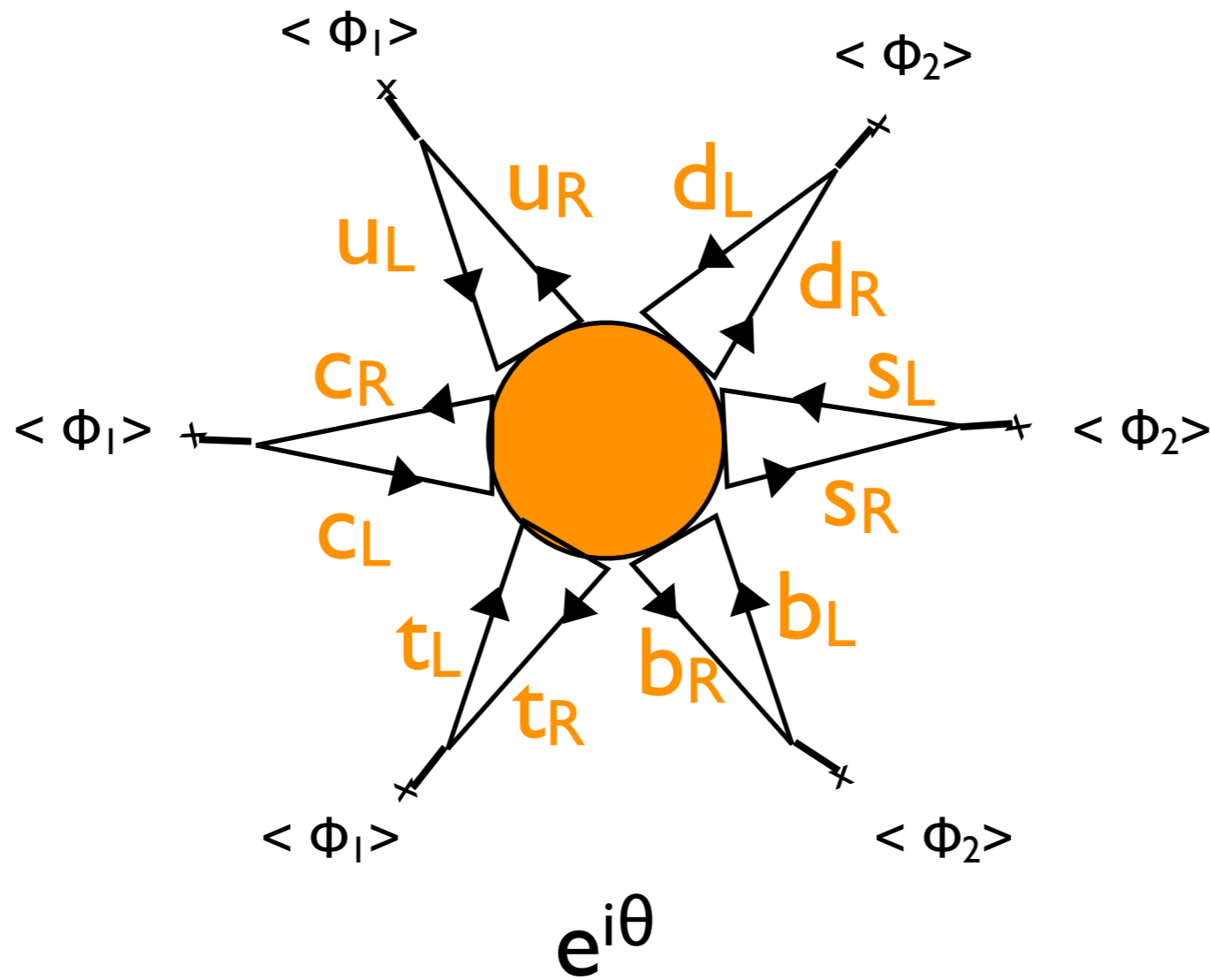
The coefficient must be small, or fine-tuned with other contributions, in order that the vacuum expectation values of Φ_1 and Φ_2 remain at the weak scale.

Experimental Constraints

Pushing Up F

As we've seen, the couplings of the axion are pretty well pinned down, given F .

Likewise for the mass:



$$V(\alpha) \sim -\cos \alpha \Lambda_{\text{QCD}}^4$$

$$m_a^2 \sim \frac{1}{F^2} \frac{d^2 V(\alpha)}{d\alpha^2} \sim \frac{\Lambda_{\text{QCD}}^4}{F^2}$$

Thus for $F = 10^3 \text{ GeV}$, $m_a \sim 10^4 \text{ eV}$;

For $F = 10^{12} \text{ GeV}$, $m_a \sim 10^{-5} \text{ eV} \rightarrow (2 \text{ cm.})^{-1}$;

... and so forth.

Several types of experiments and observations have been used to search for axions, and constrain F :

direct searches at accelerators

stellar cooling

long-range forces

conversion/reconversion of photons

emission from the Sun

Together, these probably force $F \geq 10^9$ GeV
or so.

To go further, we must consider the cosmological consequences of axions.

How Axions Affect Cosmology

And How They Might Be Detected

Cosmic Evolution of Scalar Fields

Celestial Driven Harmonic Oscillators

$$\mathcal{L} = \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$$

$$g_{00} = 1; \quad g_{ij} = -a(t)^2 \delta_{ij}; \quad \sqrt{g} = a^3$$

$$(\sqrt{g} \dot{\phi} g^{00})' = -\sqrt{g} \frac{\delta V}{\delta \phi}$$

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + m^2(T(t)) \phi = 0$$

cosmic viscosity

effective mass

When $3 \dot{a}/a \gg m$, the field is stuck.

After entering the adiabatic regime:

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + m^2(T(t))\phi = 0$$

$$\phi(t) \approx A(t) e^{i \int^t d\tau m(\tau)} \quad \text{adiabatic ansatz}$$

$$2\dot{A}m + A\dot{m} + 3 \frac{\dot{a}}{a} \dot{A}m = 0 \quad \text{“out of phase” terms}$$

$$(a^3 A^2 m)' = 0 \quad \text{adiabatic invariant}$$

Classic Axion Cosmology

A Dark Matter Candidate

The axion field is established at the PQ transition, $\langle \phi \rangle = F e^{i\theta}$.

It stores energy, due to its initial misalignment, roughly proportional to $F \sin^2\theta_0$.

ρ in play

dilution of "particles"

adiabatic damping



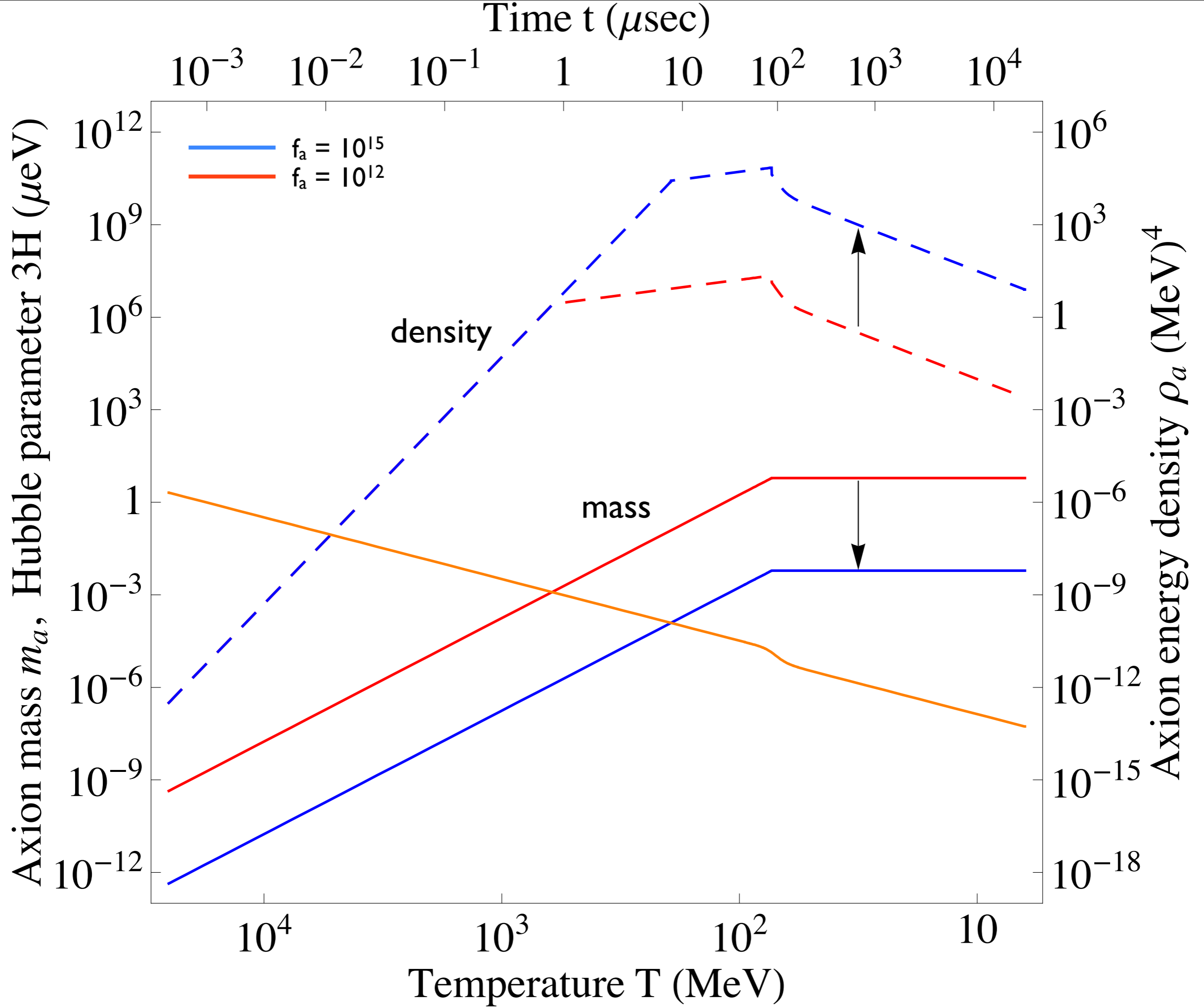
$$\Lambda^4$$

$$\left(\frac{T}{\Lambda}\right)^3$$

$$\frac{H_\Lambda}{m_a}$$

$$(T \propto R^{-1})$$

$$\frac{H_\Lambda}{m_a} = \frac{\Lambda^2}{M_{\text{Pl}}} = \frac{F}{M_{\text{Pl}}}$$



If no inflation occurs after the PQ transition, then the correlation length, which is no larger than the horizon at the transition, corresponds to a very small length in the present universe.

We therefore average over $\sin^2\theta_0$.

$F \sim 10^{12}$ GeV corresponds to the observed dark matter density.

This has usually been regarded as the default axion cosmology. A cosmic axion background with $F = 10^{12}$ GeV might be detectable, in difficult experiments.

Searches are ongoing, exploiting axion-photon conversion in magnetic field.

Inflationary Axion Cosmology

Expanded Horizons and a Multiverse

If inflation occurs after the PQ transition, things are very different.

Then the correlated volume inflates to include the entire presently observed universe, so we shouldn't average.

$F > 10^{12}$ GeV can be accommodated, with “atypically” small $\sin^2\theta_0$.

In this scenario, most of the multiverse is overwhelmingly axion-dominated, and inhospitable for the emergence of complex structure, let alone observers.

Selection effects must be considered.
(Linde, 1988).

θ_0 controls the dark matter density, but it has little or no effect on anything else. So we know what the prior measure is. (Namely, $d\theta_0$ for θ_0 , $\sin^2\theta_0 d\theta_0$ for ρ_{DM}/ρ_b .)

We do not have to get embroiled in questions of baby universe nucleation ...

... nor, for that matter: unification, supersymmetry, landscape artistry, ...

The underlying theory may be right, or it may be wrong, but it is hard to imagine a clearer case for applying anthropic reasoning.

Tegmark, Aguirre, Rees, FW astro-ph/
0511774

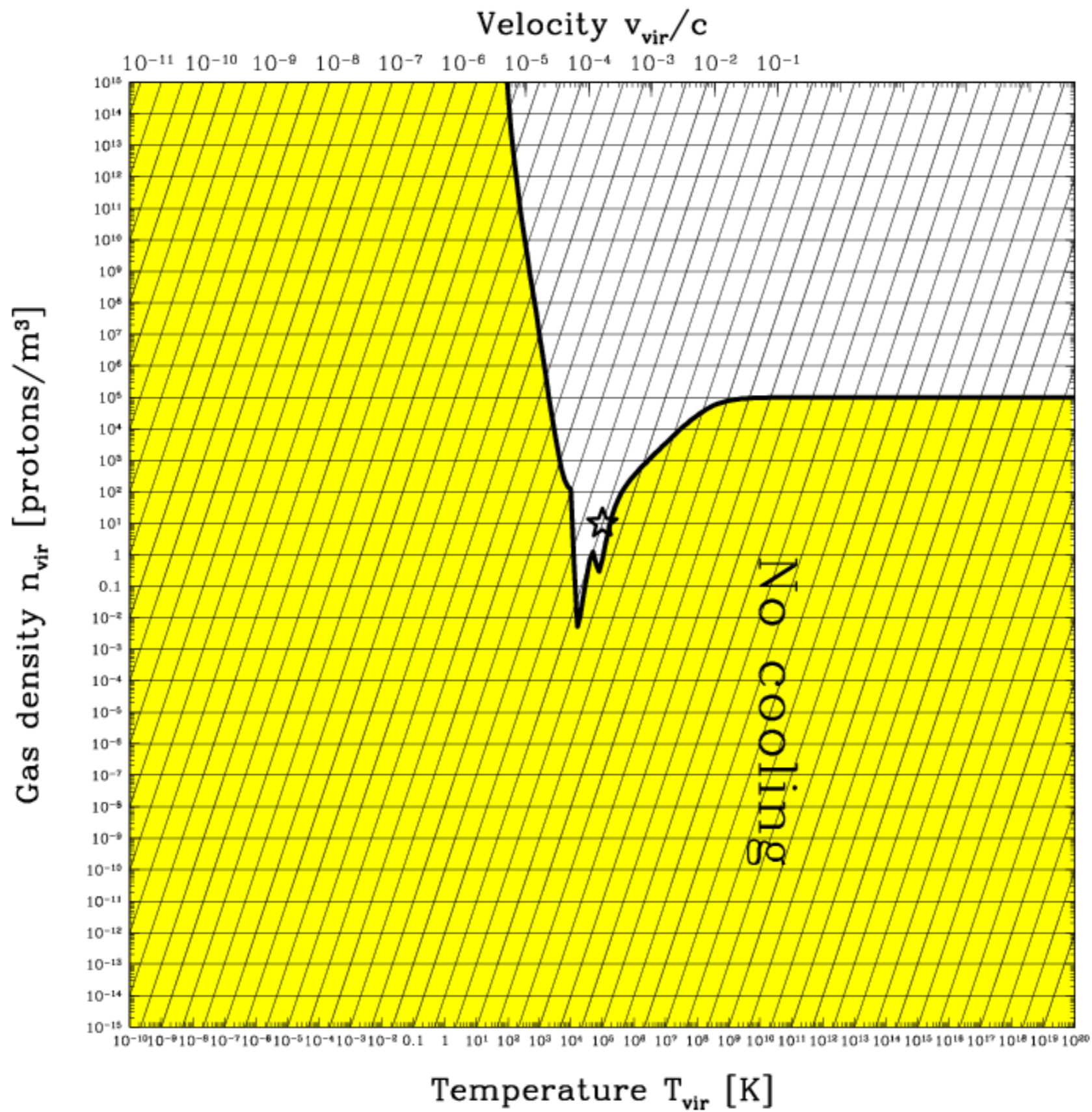
Interlude: The Fragility of Life

Selection Effect and Cosmological Parameters

Lots of things can go wrong when you try to make nice solar systems, starting from small seed fluctuations.

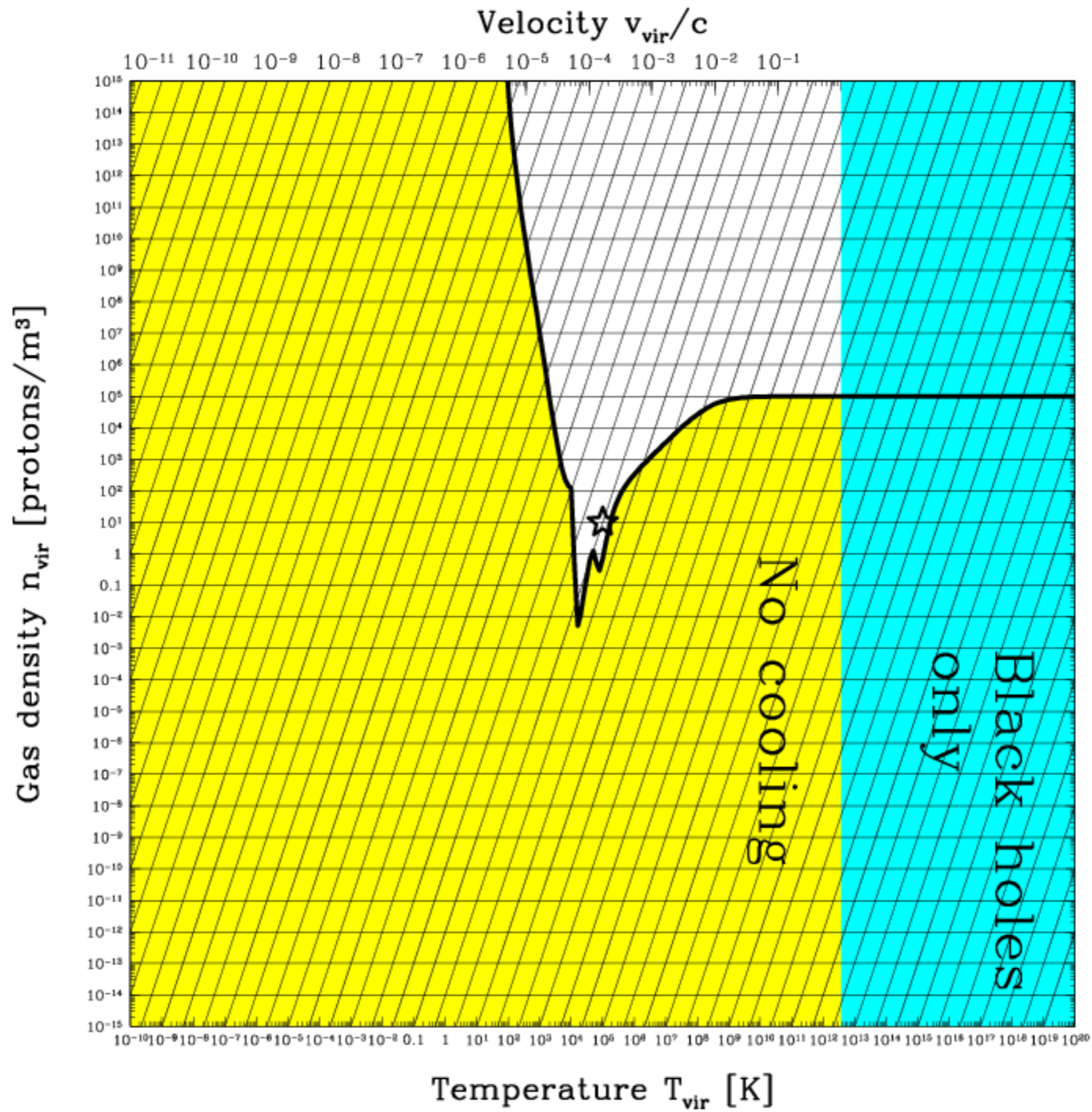
The (normal) matter might fail to cool, so it sloshes around and remains diffuse:

density ↑
time ↓
size ↓



Your fluctuations might collapse into black holes:

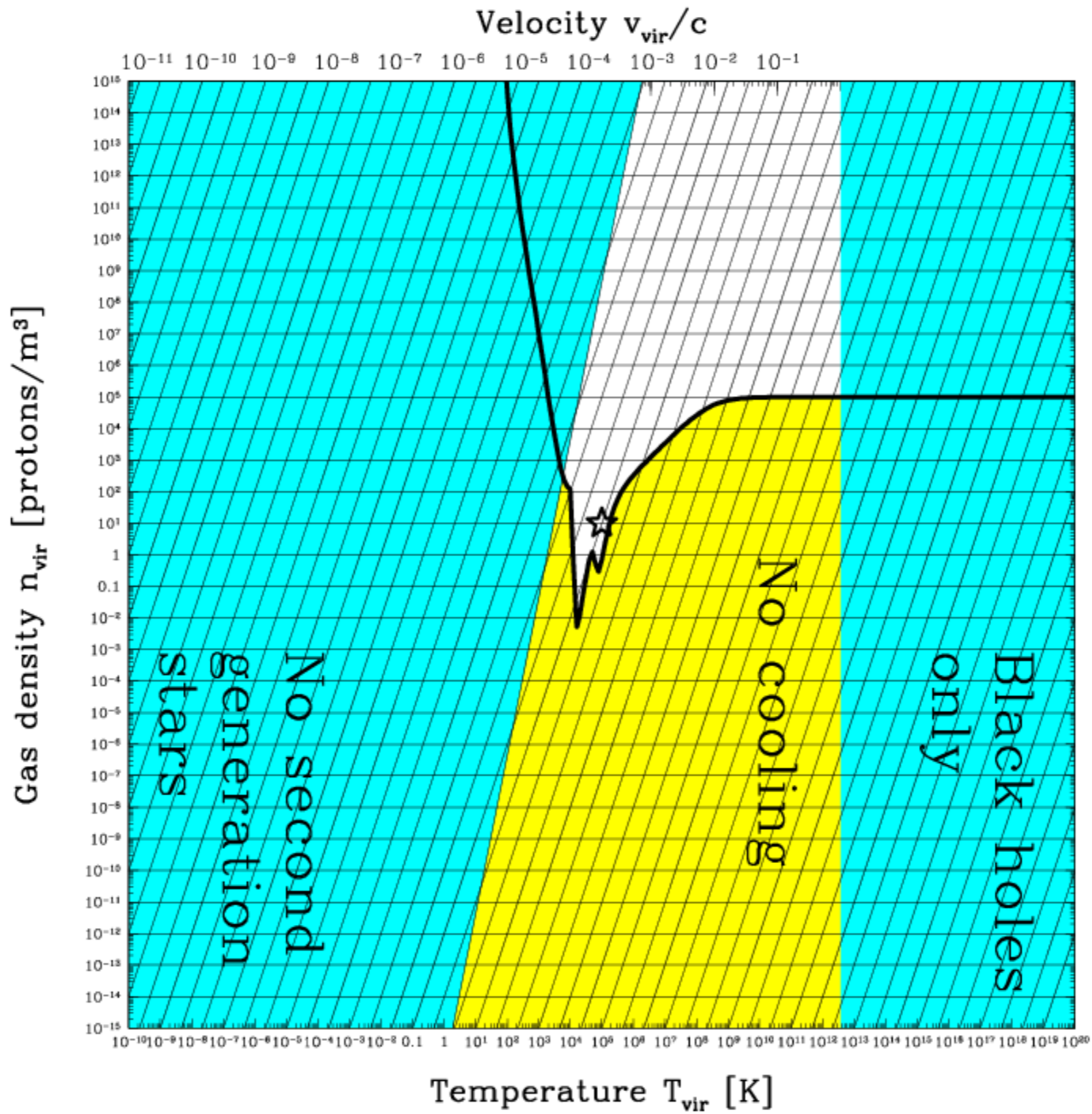
density ↑
time ↓
size ↓



contrast →

The matter might get swept out by the first supernovae:

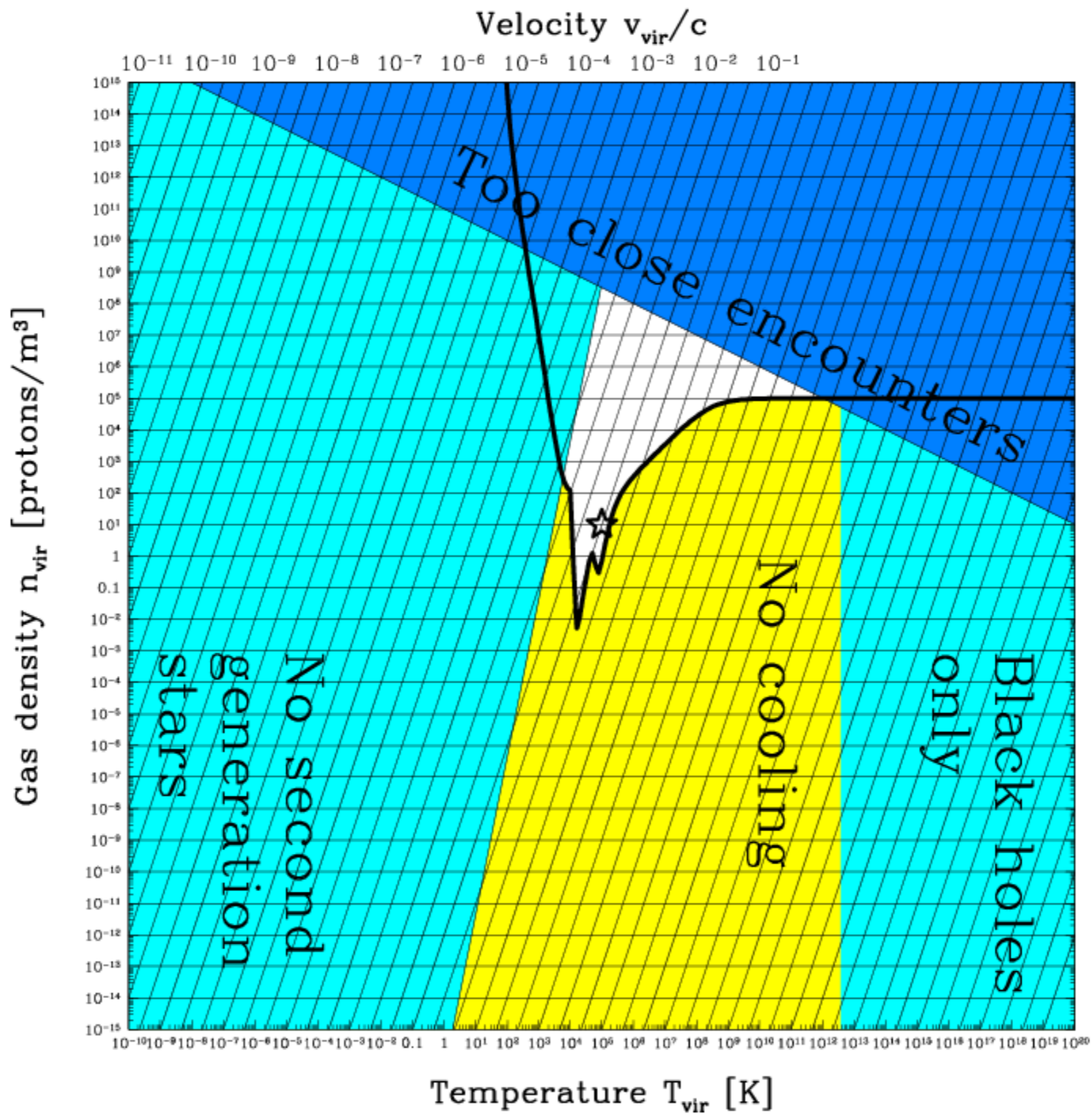
density \uparrow
time \downarrow
size \downarrow



contrast \rightarrow

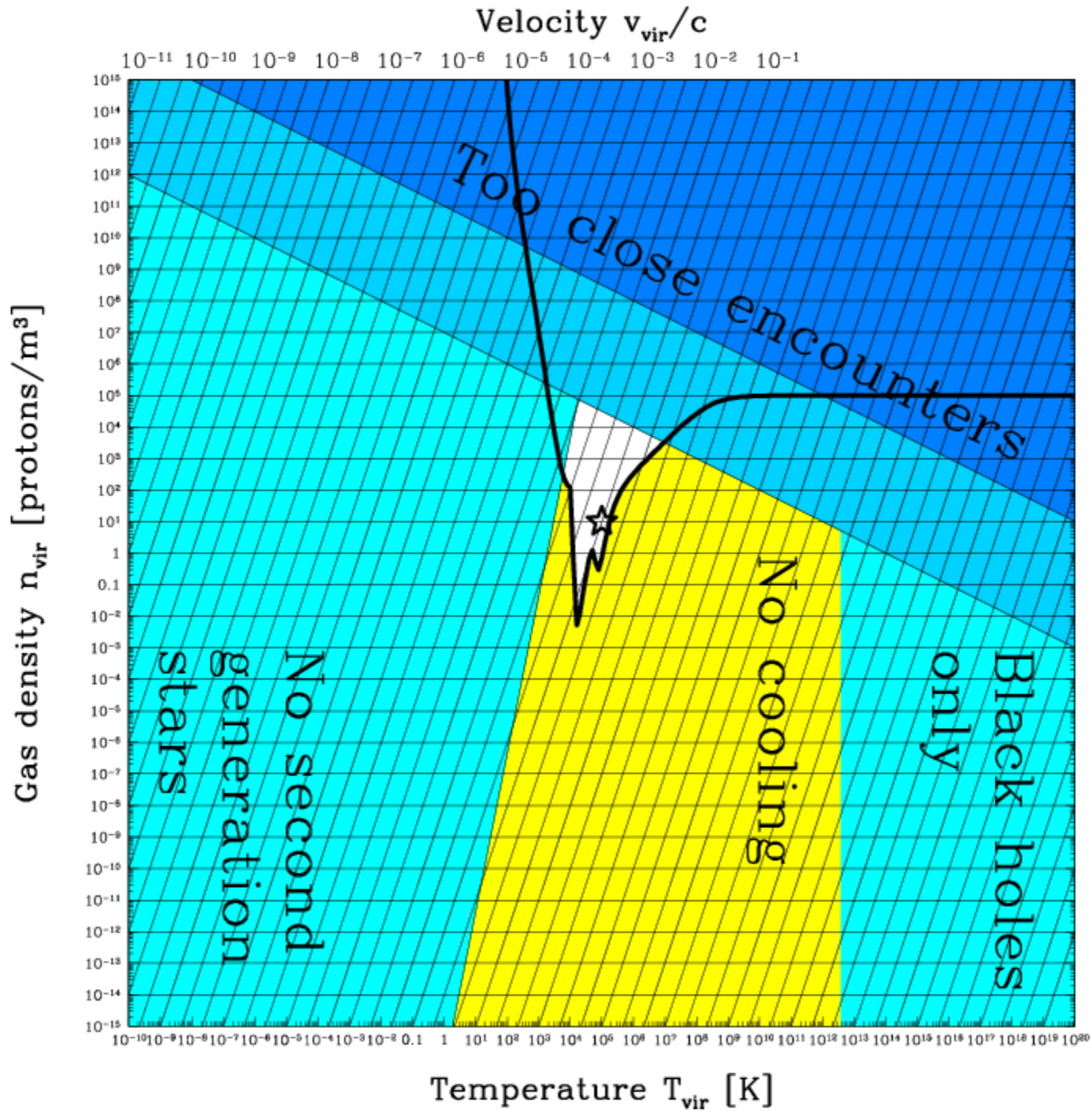
**There might be no safe haven from
disruptive encounters:**

density ↑
time ↓
size ↓



contrast →

density ↑
time ↓
size ↓

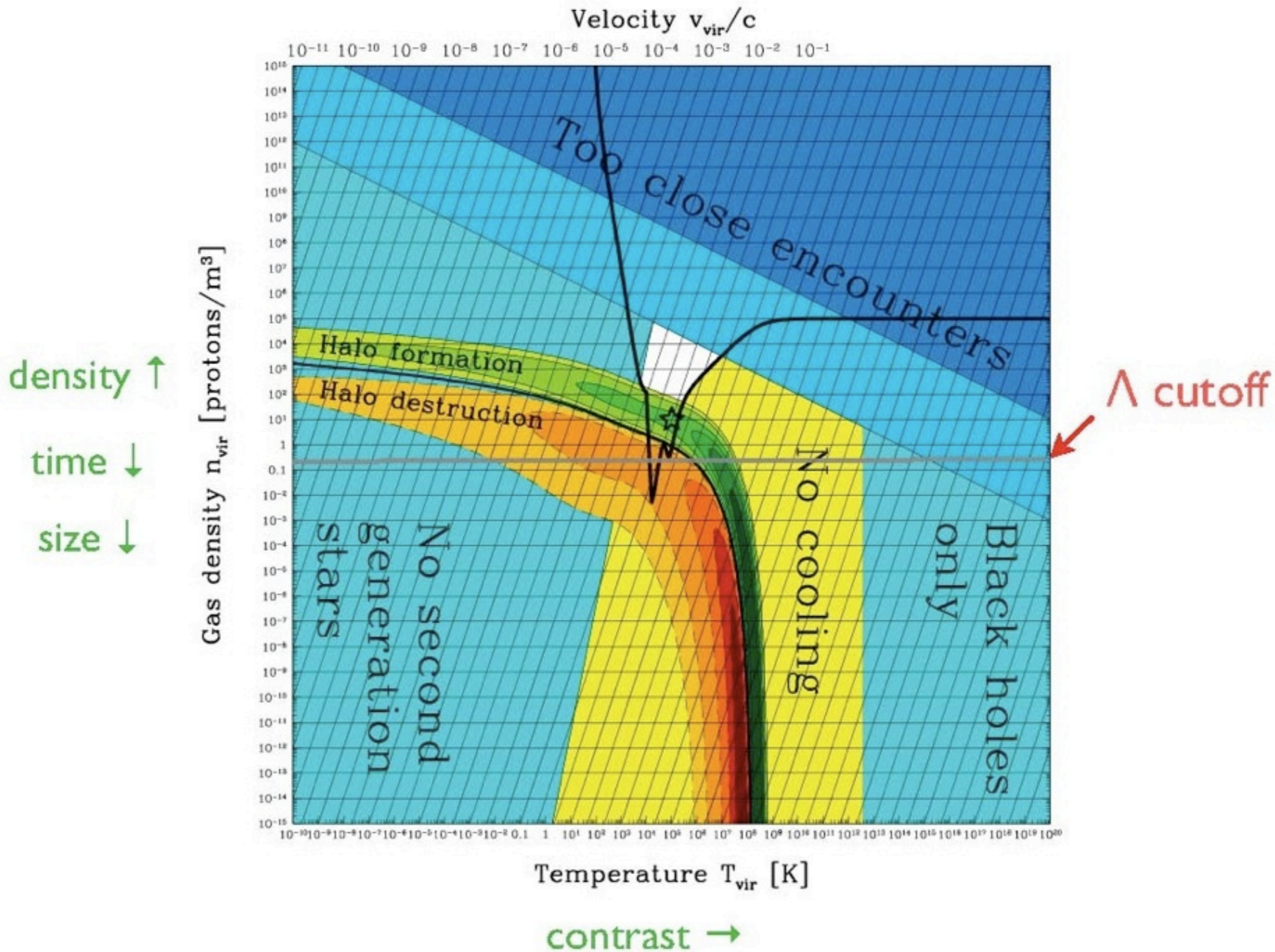


contrast →

Back to cosmology ...

We can overlay the fate map of seeds with the census of seeds we get from primordial fluctuations.

Here is what we get with the standard fluctuation spectrum and the observed dark matter density:

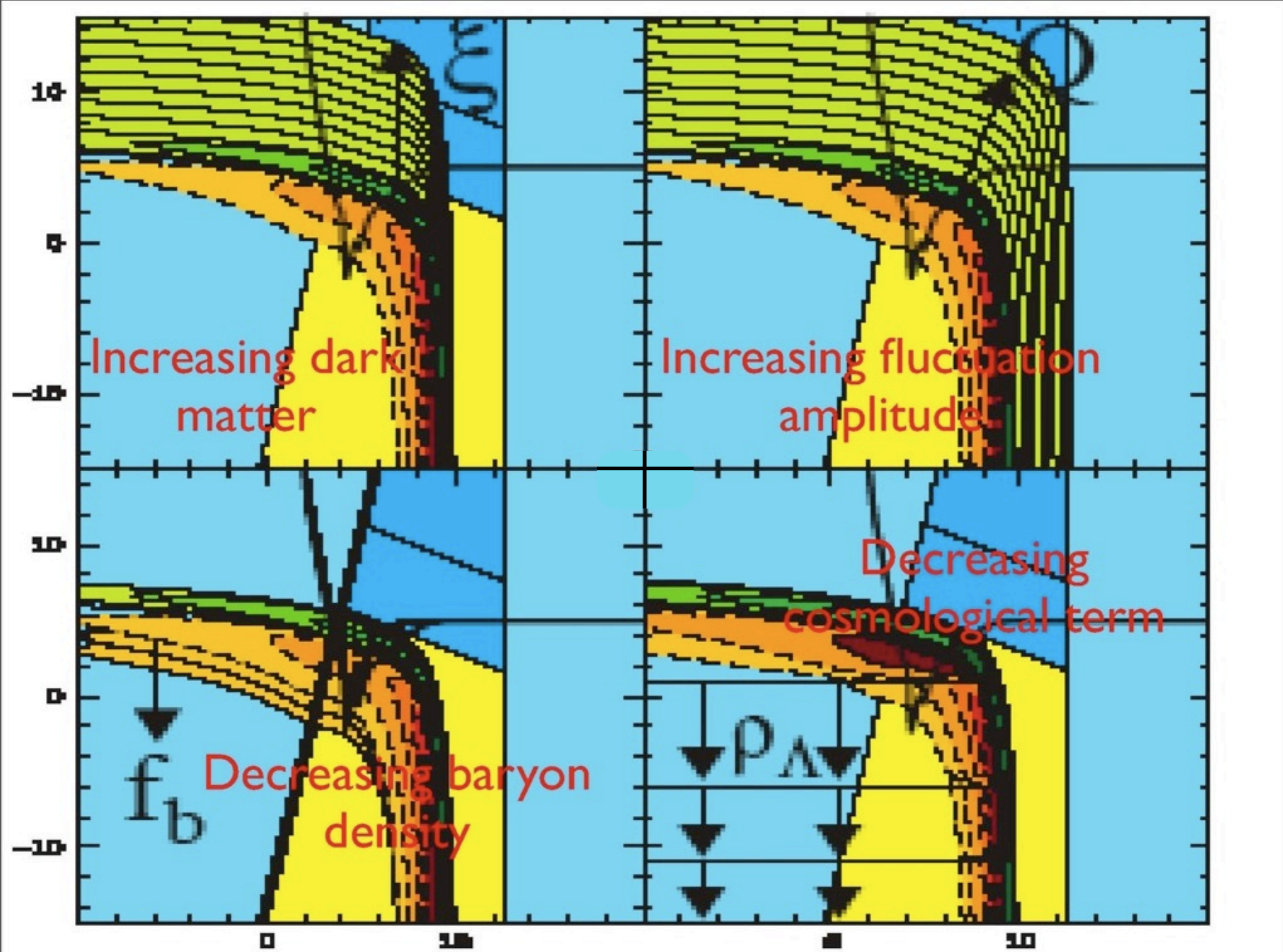


When dark energy starts to dominate, and exponential expansion kicks in, growth of new structure is inhibited. This provides the Λ_{cosmo} cutoff.

These calculations provide a semi-quantitative explanation of the characteristic size of galaxies.

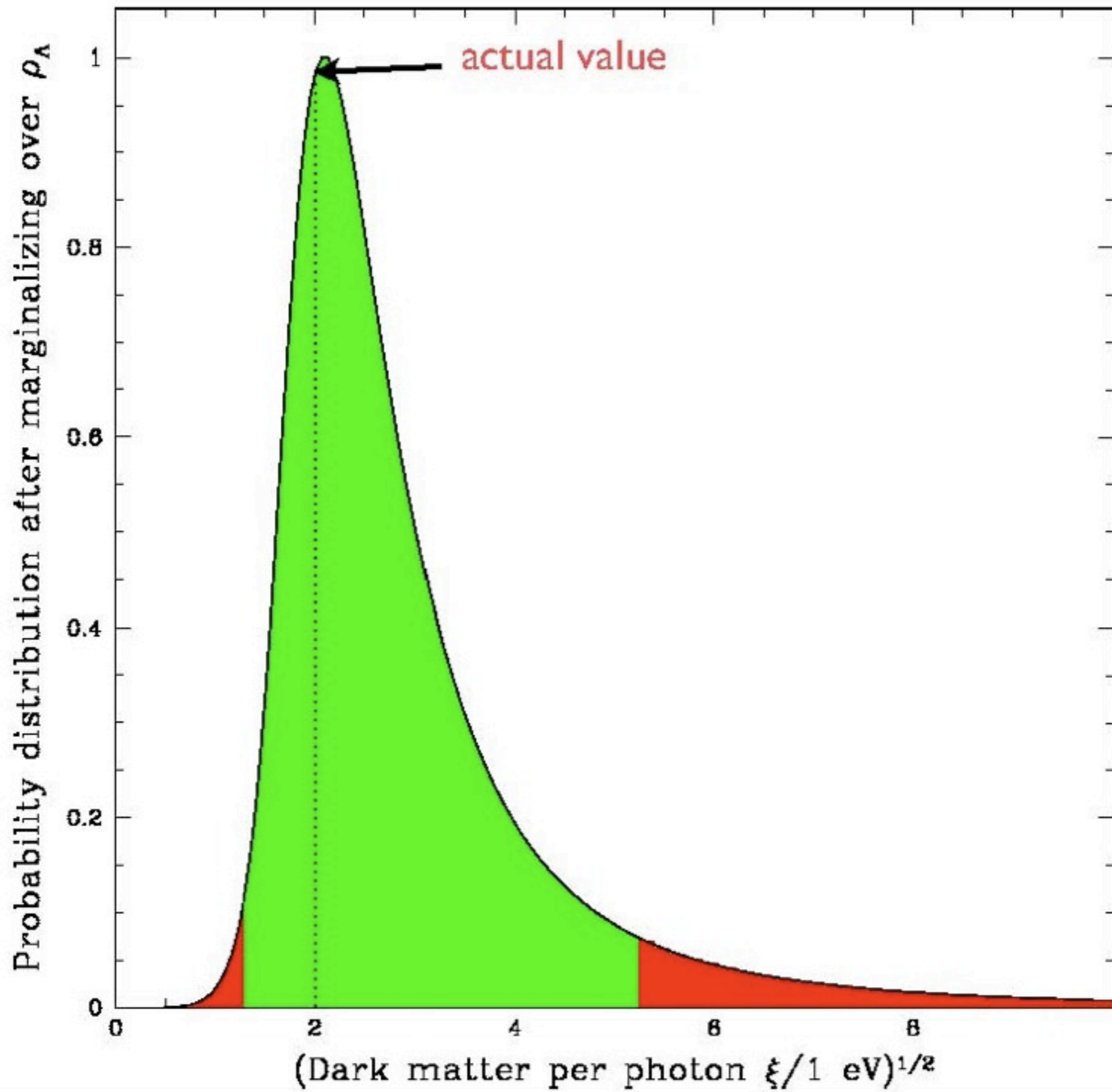
So far what what we've done is entirely conventional astrophysics.

With our confidence reinforced, we now consider the effect of varying parameters that govern the primordial fluctuations:



We implement selection bias by calculating probability distributions *per baryon in the user-friendly region* (not per unit volume).

Here is the θ_0 distribution, translated into dark matter density:



This is a striking result, I think.

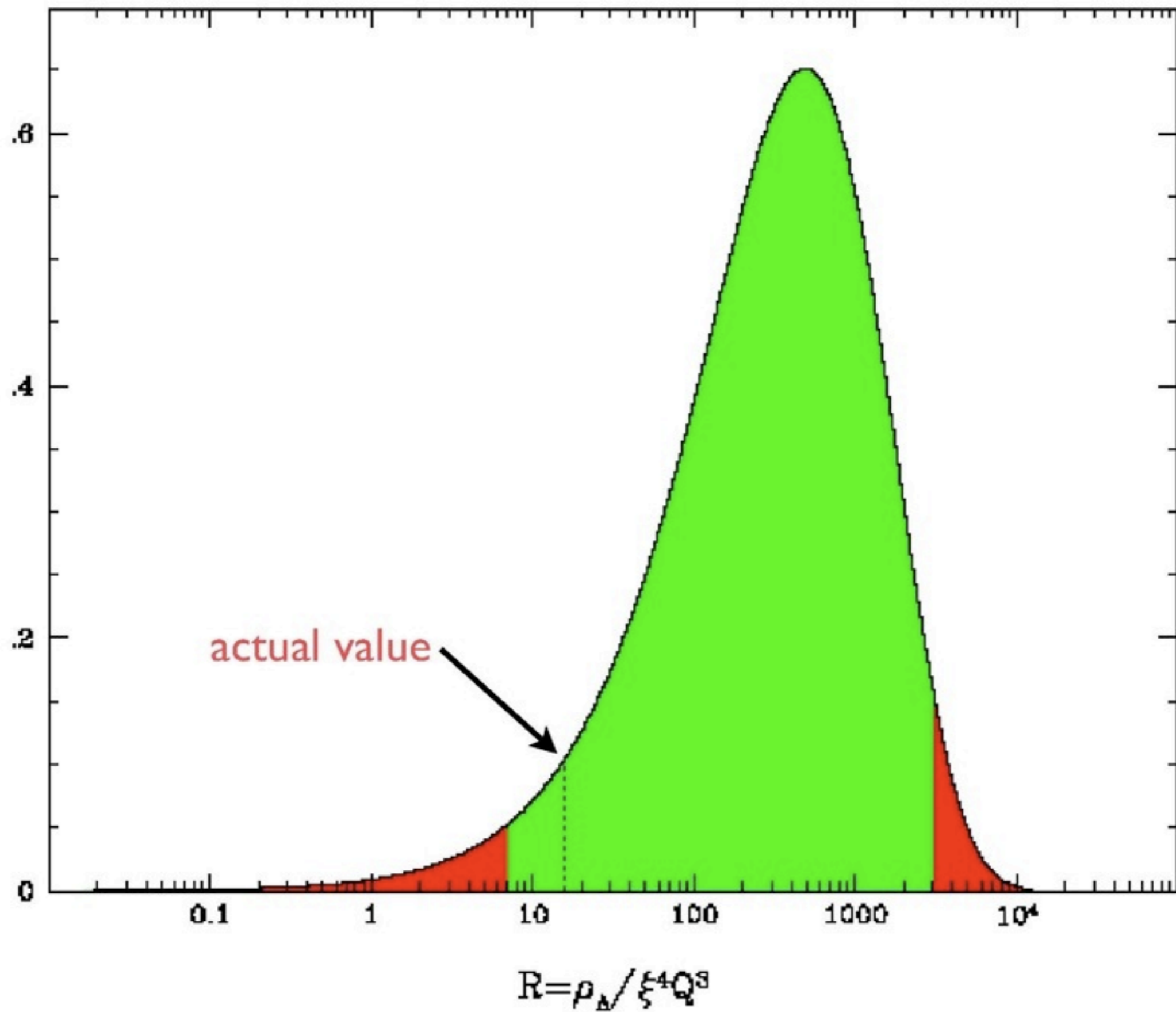
The scenario with inflation after the PQ transition also removes some annoying difficulties of the traditional alternative, including the need to introduce a new mass scale in fundamental physics.

Axion string and domain walls also disappear.

We can apply similar reasoning to the cosmological term (= dark energy).

Here is the ρ_Λ distribution, given a flat prior, and **“holding everything else fixed”***.

*Of course, this is ill-defined conceptually: e.g., should we hold Q fixed ... or $Q \rho_\Lambda$... or $Q \rho_\Lambda P$?



The result is suggestive, but its foundation is very insecure.

Isocurvature Fluctuations

An Incisive Probe

A canonically normalized boson field - graviton or axion - acquires fluctuations of amplitude $T_{GH} \sim \Lambda_{\text{infl.}}^2/M_{\text{Pl}}$.

For axions, this translates into jitter in θ_0 , and thus ultimately into isocurvature density fluctuations.

Constraints on isocurvature fluctuations translate into constraints on $\Lambda_{\text{infl.}}$, and thus on the gravity wave background.

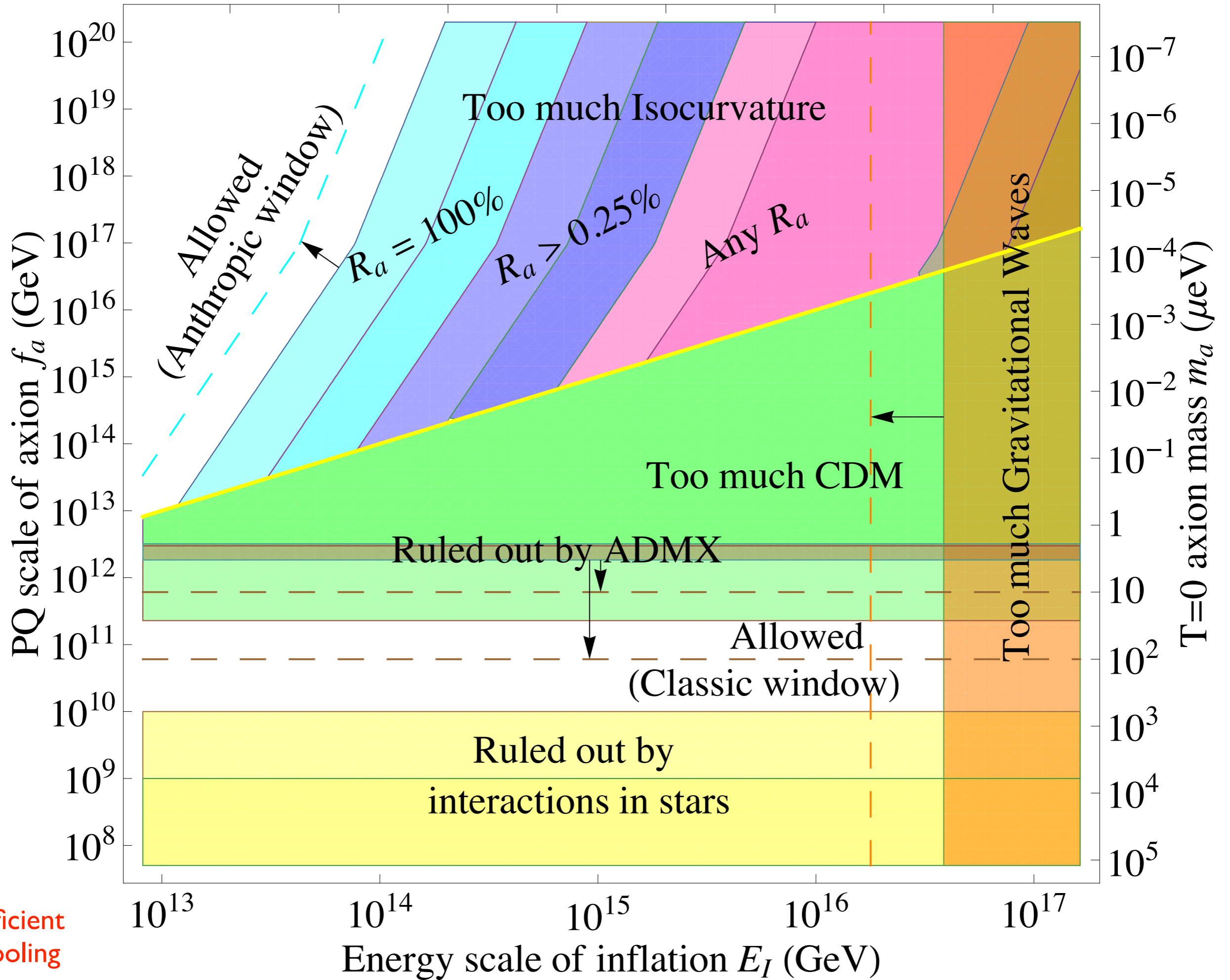
For axions as dark matter

$$(f_a/N)\theta_i^2 \sim 10^{12} \text{ GeV}$$

$$\alpha_a \approx \frac{2}{25\pi^2} \frac{H_I^2}{\bar{f} \cdot 10^{12} \text{ GeV}}$$

or, using $H_I = 5\pi Q_t \bar{m}_{\text{Pl}}$,

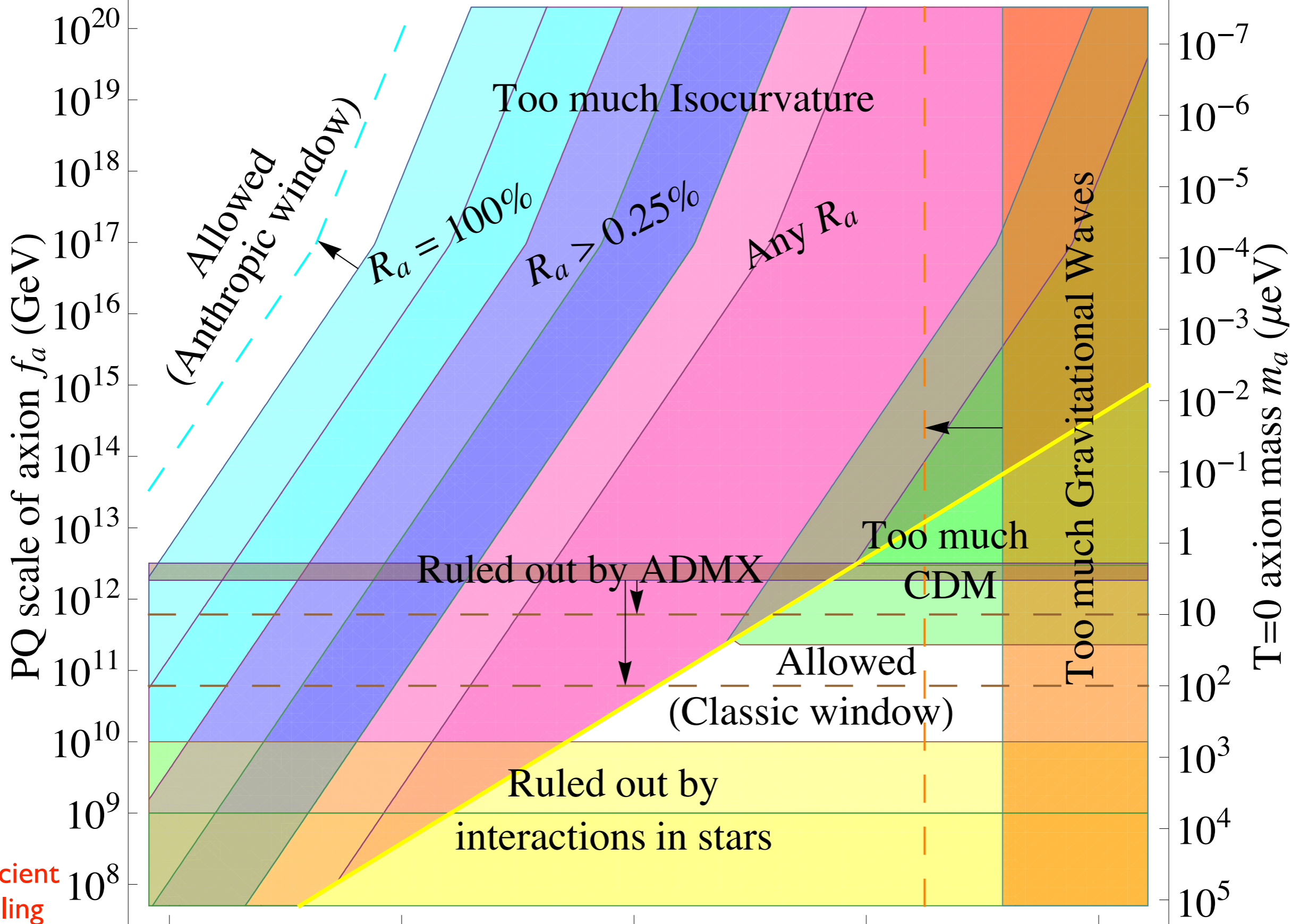
$$\alpha_a \approx Q_t^2 \frac{2 \bar{m}_{\text{Pl}}^2}{\bar{f} \cdot 10^{12} \text{ GeV}}$$



efficient cooling

Hubble scale of inflation H_I (GeV)

10^8 10^9 10^{10} 10^{11} 10^{12} 10^{13} 10^{14} 10^{15}



Thus the inflationary axion cosmology would be falsified, were we to see a significant gravitational wave background without a larger isocurvature background.

It could be “truthified” if we still have a dark matter after LHC (+ILC?); through details of the dark matter distribution; or if we discover isocurvature fluctuations.

Even if SUSY and a dark matter candidate are found at LHC, it will be important to pin its properties down and calculate its cosmological production. Axions will happily (and naturally) rectify any deficit.

Grand Summary

The standard model account of T violation is profound and successful, but conceptually flawed.

We can improve the situation by expanding the equations in a fairly simple way, to support additional symmetry.

The expanded equations predict the existence of a new particle, the axion, with remarkable properties.

Experiments constrain the key parameter of axion physics, F , to large mass values (well beyond weak or LHC scales).

Axions are then almost forced to be important for cosmology, and contribute significantly to dark matter, if they exist at all.

[Now follow some slides that were not used in the lectures, for future development.]

Other Puzzles of T Symmetry

Brief Mentions

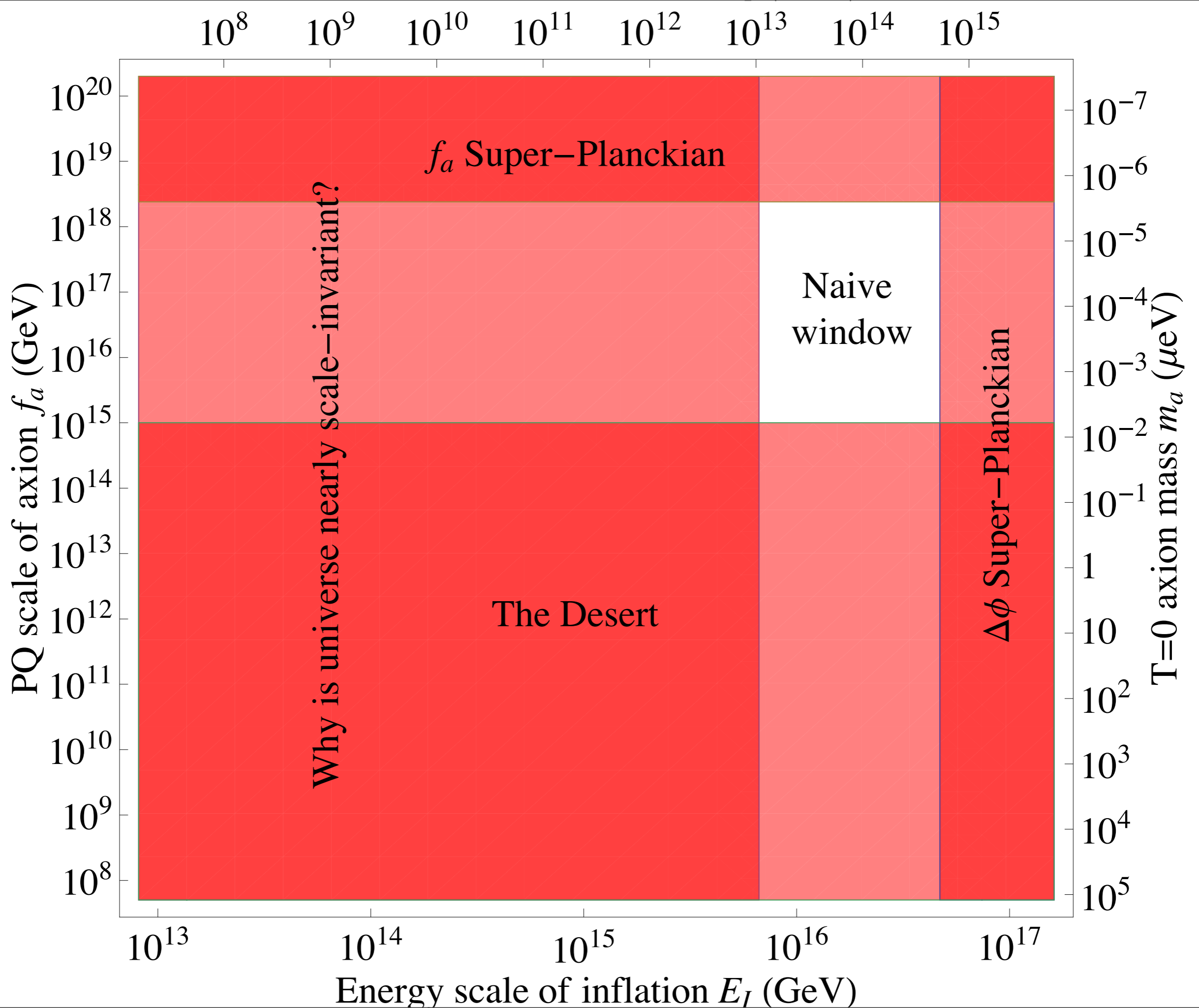
I have not discussed leptons. Neutrino oscillations offer access to another possible CP violating angle, broadly similar to the one for quarks.

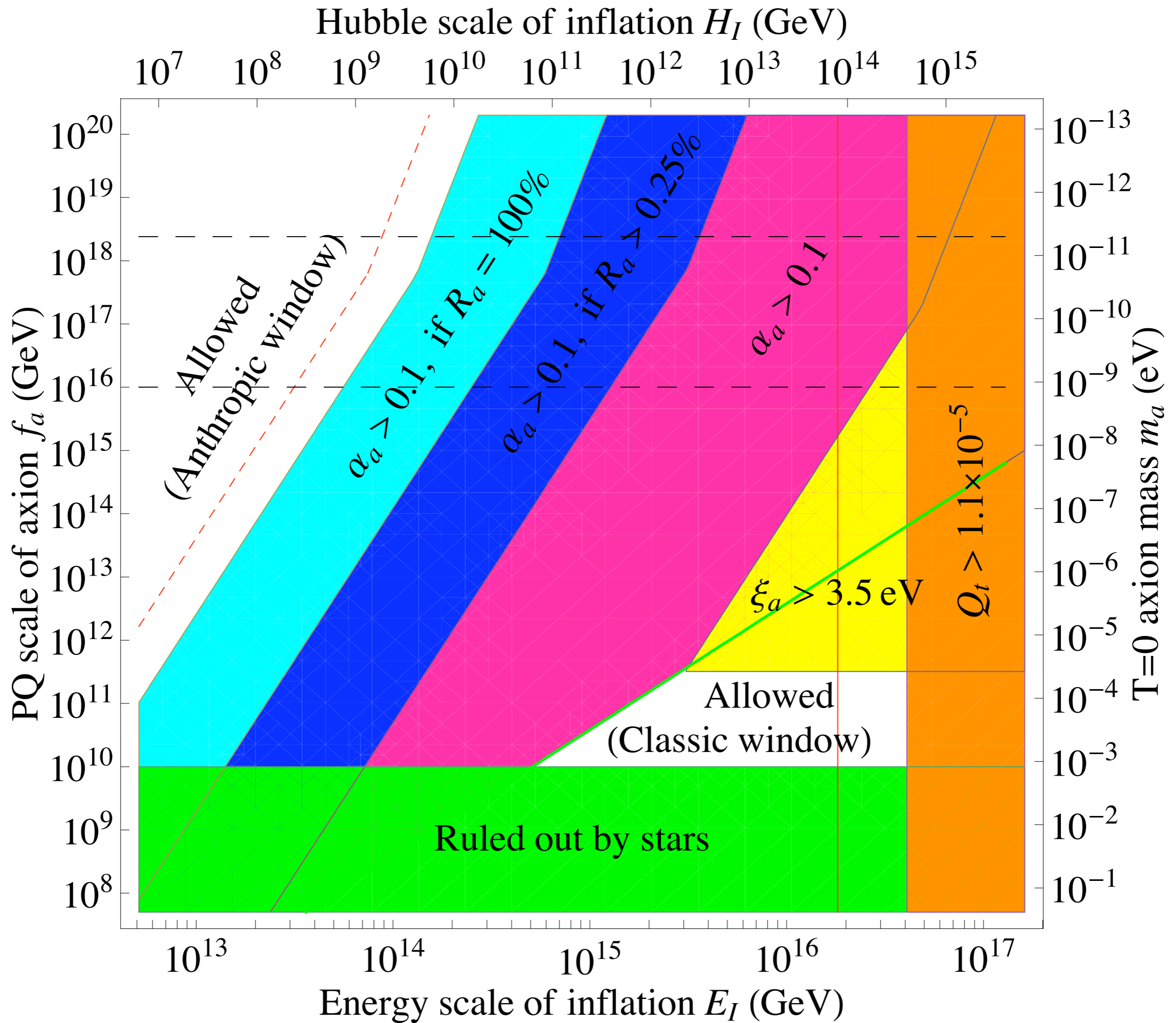
In models of low-energy supersymmetry, there are many potential additional contributions to CP violation, both flavor-violating and flavor-conserving. So far there is no sign of them.

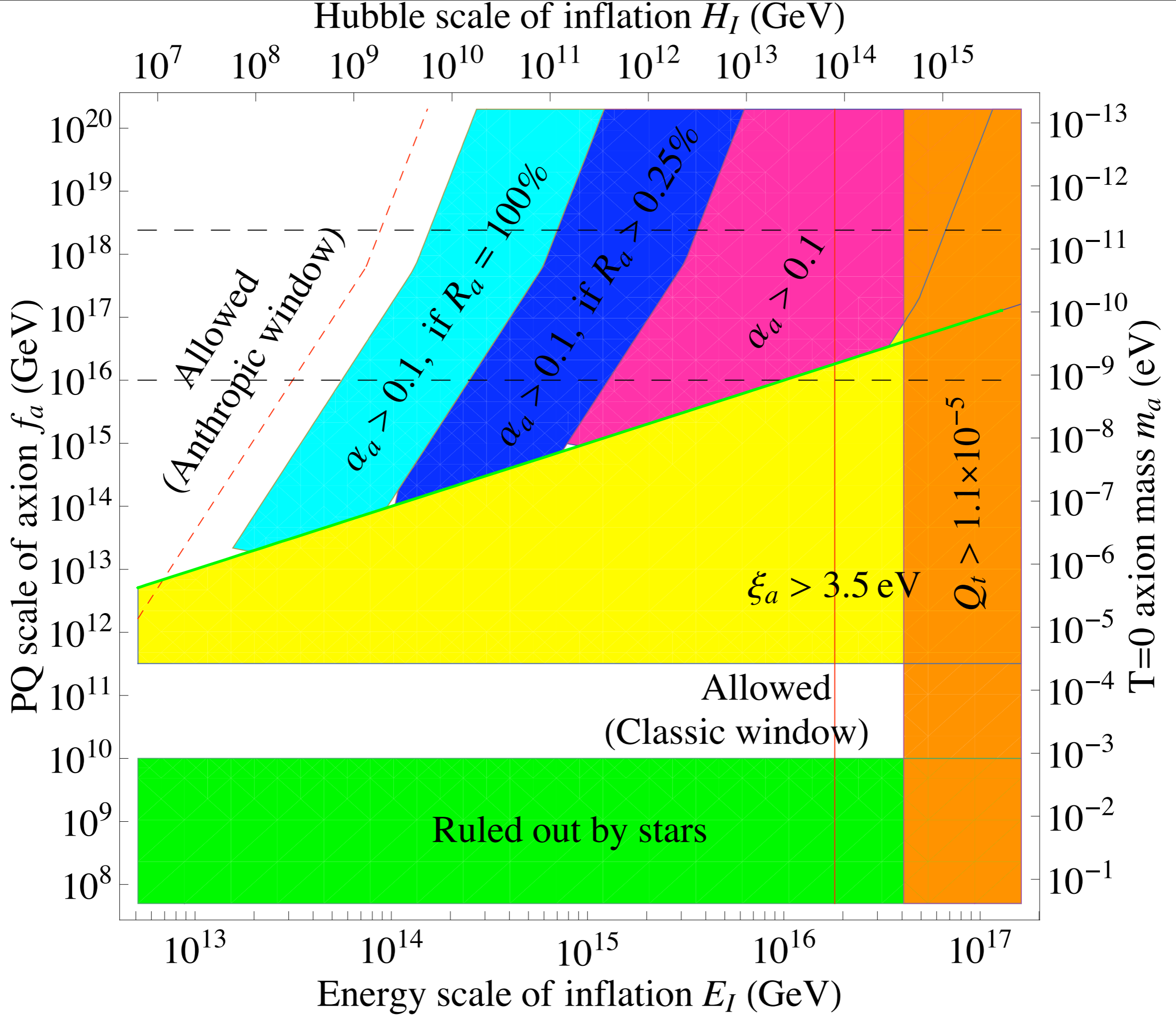
CP violation is a necessary ingredient of baryogenesis. (Since $\langle B \rangle$ is CP odd.) The basic interactions responsible for baryogenesis in the early universe might not be experimentally accessible today, however.

More About EDMs

A Low-Energy Probe of High-Energy Physics







$$\alpha_a \equiv \frac{\langle (\delta T/T)_{\text{iso}}^2 \rangle}{\langle (\delta T/T)_{\text{tot}}^2 \rangle}$$

$$\approx \frac{8}{25} \frac{(\xi_a/\xi_m)^2}{\langle (\delta T/T)_{\text{tot}}^2 \rangle} \sigma_\theta^2 \frac{2\theta_i^2 + \sigma_\theta^2}{(\theta_i^2 + \sigma_\theta^2)^2}$$

$$\sigma_\theta = \frac{H_I}{2\pi(f_a/N)}$$