Consistent Histories: Basic Concepts

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Abstract

This is a brief introduction to the consistent histories approach to quantum mechanics. Aside from some orienting commentary, it is a proper – but also dense – subset of the first 11 chapters of Griffiths "Consistent Quantum Theory".

1. *Physical Properties and Logical Operations*

In classical mechanics, the state of a system is specified by a point in phase space. In quantum mechanics, it is specified by a ray in Hilbert space, or equivalently by the one-dimensional projection operator

$$P_\psi \; = \; |\psi\rangle \langle\psi| \; \equiv \; [\psi] \tag{1}$$

Note that P is an Hermitean operator with $P^2 = P$

In classical mechanics, a (binary, yes-or-no) *property* $\mathcal{P}$ of states is specified by an indicator or characteristic function $\chi_{\mathcal{P}}$ on phase space, which takes the value 1 on the set of states for which the property is true, and 0 where it is false.

In quantum mechanics, a property of states is a linear subspace of Hilbert space, or equivalently the associated projection operator Q (not necessarily one-dimensional, of course). So Q is Hermitean and $Q^2 = Q$. We say the state $|\psi\rangle$ has the property $\mathcal{P}_Q$ associated with Q if $Q|\psi\rangle \; = \; |\psi\rangle$, and that it does not have the property Q if $Q|\psi\rangle \; = \; 0$. If neither of these conditions holds, then the property $\mathcal{P}_Q$ is undefined on $|\psi\rangle$. This possibility has no classical analogue.

Note that the states which either do or do not have the property $\mathcal{P}_Q$ are precisely the eigenstates of Q. An equivalent condition is

$$Q[\psi] \;=\; [\psi]Q \tag{2}$$

i.e. that the property $\mathcal{P}_Q$ is defined on $|\psi\rangle$ if and only if Q commutes with $[\psi]$.

For any quantum property Q[1], we can implement NOT-Q by $1 - Q$. With this definition, NOT-Q has the properties that we expect in logic, with the added feature that it is undefined precisely when Q is.

If Q, R are two *commuting* properties, $QR = RQ$, then it makes sense to define

$$Q \text{ AND } R \;\equiv\; Q \wedge R \;=\; QR \tag{3}$$

Indeed, QR is a projection operator, and it projects on states that are in both of the "true" subspaces for Q and R.

On the other hand if Q does not commute with R then QR is neither Hermitean, nor equal to its square. There is no obvious way to make sense of the notion that Q and R are both true. An example of this situation, which shows its hopelessness, is $Q = \sigma_3$, $R = \sigma_1$ acting on the 2-dimensional Hilbert space of a spin-$\frac{1}{2}$ system. If we want to maintain as much of classical logic as we can in the quantum domain, it seems most fruitful to declare that when $QR \neq RQ$ those properties are *incompatible*. They cannot both be ascribed to the same system at the same time.

For compatible properties we can define logical OR according to

$$\begin{aligned} Q \text{ OR } R &= \text{NOT}\,\Big((\text{NOT}\,Q) \text{ AND } (\text{NOT}\,R)\Big) \\ &= 1 - \Big((1-Q)(1-R)\Big) \;=\; Q + R - QR \end{aligned} \tag{4}$$

and so forth. All the definitions and results of propositional calculus carry over, as long as we consider algebras containing commuting properties only.

Note that *meaningless* is quite different from *false*. Indeed, "false" is quite meaning*ful*.

[1]Henceforward I'll identify quantum properties with the operators that implement them, when no confusion can arise.

Similarly, *incompatible* is quite different from *contradictory.* P and Q are contradictory properties if $PQ = 0$, and in that case they certainly commute, so they are compatible!

[discussion points: partial compatibility, quantum logic (Birkhoff-von Neumann)]

One can have two rich and internally compatible sets of properties $\{P_j\}$, $\{Q_k\}$ that are mutually incompatible. Then each is a valid way to describe our system, but they cannot be applied simultaneously. The classic example in quantum mechanics is position and momentum. We can construct appropriate "properties", in the present sense, by projection on intervals (or more general subsets of the line). For example, the operator that takes a position-space wave function $\psi(x)$ for $-\infty < x < \infty$ and projects it onto the function

$$\begin{aligned} \psi_{[a,b]}(x) &= 0,\ x < a \\ \psi_{[a,b]}(x) &= \psi(x), a \leq x \leq b \\ \psi_{[a,b]}(x) &= 0,\ b < x \end{aligned} \tag{5}$$

is a projection operator. The projection operators in position space all commute, and so are compatible. Similarly, we can form an algebra of compatible projections in momentum space. But these two algebras are mutually incompatible.

The possibility of rich and individually valid, but mutually incompatible, descriptions is the essence of Bohr's beloved *complementarity.* Its mind-expanding significance ramifies far beyond quantum mechanics. Kant's interpretation of the physical versus moral description of human beings and (transcendence of) the mind-body "problem" – psychological versus physical descriptions of mental/brain states – are, I believe, profound examples. They might seem more *ad hoc* and far-fetched, had we not the very concrete, quantitative, and successful model of quantum complementarity before us.

2. *Physical Variables and Probabilities*

   In classical physics one can define a sample space of mutually exclusive properties by decomposing the identity function in phase space into a sum of characteristic functions. Alternatively, one carves phase space into disjoint sets whose union is the whole space. A random variable on this sample space is an assignment of probabilities p_α, with $0 \leq p_\alpha \leq 1$, $\sum\limits_\alpha p_\alpha = 1$, to the sets ... with probability 0 for the null set.

In quantum physics the appropriate concept is a decomposition of the identity function in Hilbert space into a sum of mutually commuting projection operators:

$$
\begin{aligned}
1 &= \sum_\alpha P_\alpha \\
P_\alpha^\dagger &= P_\alpha \\
P_\alpha P_\beta &= \delta_{\alpha\beta} P_\alpha
\end{aligned}
\tag{6}
$$

Alternatively, we can view this as a decomposition of Hilbert space into mutually orthogonal linear subspaces.

Each P_α can defines a property of our system, as discussed above. The properties are compatible. We can assign probabilities p_α to the different properties, and the usual rules of probability calculus will apply. One can, of course, consider different decompositions of the identity. These correspond to asking different questions about the system. If all the projection operators in two decompositions $\{P_\alpha\}, \{Q_\beta\}$ commute then we can define a common *refinement* through the decomposition using all the products $P_\alpha Q_\beta$. In this refinement, we can recover the P_α by summing $P_\alpha Q_\beta$ over α, and so forth, so nothing has been lost. In the refinement, we can ask both the old questions and more refined "AND" questions.

On the other hand one can also have incompatible decompositions. They correspond to complementary questions, and cannot be combined.

For any Hermitean operator A, one can introduce the a sample space based on its eigenspaces. The set of properties "A has the value $v_\alpha(A)$", where $v_\alpha(A)$ runs over the eigenvalues of A, is represented (minimally) in that sample space. Commuting Hermitean operators define compatible sample spaces; non-commuting operators define incompatible sample spaces.

3. *Histories and Their Logic*

So far we have discussed properties of a system at one time. To discuss dynamics, we need to bring in the concept of histories.

We consider several times $t_i < t_1 ... < t_f$. For each time we have a Hilbert space of states H_{t_j}, and histories are defined in the tensor product $\mathcal{H}$ of those Hilbert spaces

$$
\mathcal{H} = H_{t_f} \otimes ... \otimes H_{t_1} \otimes H_{t_i} \tag{7}
$$

with the earlier times occurring to the right. A history is the choice, for each time t_j, of a property F_{t_j} in H_{t_j}[2]. (F_{t_j} can be the identity, if we allow anything to happen at t_j; if $F_{t_j} = 0$, we have the null history.) A history defines a property in the history Hilbert space, according to the projector

$$Y \;=\; F_{t_f} \otimes ... \otimes F_{t_1} \otimes F_{t_i} \tag{8}$$

It can be interpreted to mean that the system has property F_{t_j} at each time t_j.

We can take over most of our previous discussion of properties to this new context. Thus one has compatible and incompatible histories, logical operations on compatible histories, sample spaces of histories, refinements of compatible sample spaces of histories ...

4. *Born Rule: Special*

Now we bring in dynamics – to be sure, very abstractly – by assuming we are given unitary evolution operators $T(t_b, t_a)$ connecting the Hilbert spaces for any two times t_b, t_a. We assume the conditions

$$\begin{aligned} T(t_b,t_a)^{-1} &= T(t_b,t_a)^{\dagger} \;=\; T(t_a,t_b) \\ T(t_c,t_b)T(t_b,t_a) &= T(t_c,t_a) \\ T(t_a,t_a) &= 1 \end{aligned} \tag{9}$$

In many applications of quantum mechanics, we would have $T(t_b, t_a) = e^{-iH(t_b - t_a)}$, where H is the appropriate Hamiltonian.

It is instructive to formulate the Born rule for assigning probabilities of "observables" to states in our present language. Let us suppose that we know our system is in the state $|\psi\rangle$ at t_0, and we want to know whether it has property P at a later time t_1. The Born rule tells us that this occurs with probability

$$\text{Pr} \;=\; \parallel PT(t_1,t_0)|\psi\rangle \parallel^2 \tag{10}$$

involving the square of the projected evolved wave function. We can re-write this is a suggestive way as

$$\text{Pr} \;=\; \langle\psi|T(t_1,t_0)^{\dagger}PT(t_1,t_0)|\psi\rangle \;=\; \text{Tr}\left(T(t_1,t_0)^{\dagger}PT(t_1,t_0)[\psi]\right) \tag{11}$$

[2]This is slightly different from, and more restrictive than, defining a history simply as a property in the history Hilbert space. Whether the more general notion, which seems natural, is interesting or useful, I don't know

We can interpret this as connecting the unitarily evolved *property* $[\psi]$, i.e. $T(t_1, t_0)[\psi]T(t_1, t_0)^\dagger$, to the property P, with the inner product of operators defined by the trace. This is the interpretation we will run with, and generalize.

Note that if we take $t_1 = t_0$, we encounter a subtlety: the property $[\psi]$ may be, and in interesting cases usually will be, incompatible with P. In that case (and, really, in general) we should not regard $[\psi]$ as introducing a new property, but rather as a mathematical summary of our knowledge of the system. Griffiths calls it a pre-probability.

5. *Born Rule: General*

Now let us generalize Eqn. (11), so that we can discuss probabilities of histories. Suppose that we have a sample space of histories, with typical representative

$$Y^\alpha \ = \ F^\alpha_{t_f} \otimes ... \otimes F^\alpha_{t_1} \otimes F^\alpha_{t_i} \tag{12}$$

We define the *chain operator* $K(Y^\alpha)$ through[3]

$$K(Y^\alpha) \ = \ F^\alpha_{t_f} T(t_f, t_{f-1}) F^\alpha_{t_{f-1}} ... T(t_2, t_1) F^\alpha_{t_1} T(t_1, t_i) F^\alpha_{t_i} \tag{13}$$

And now we define the *weight* of history Y^α as

$$W(Y^\alpha) \ = \ \mathrm{Tr}\, K(Y^\alpha)^\dagger K(Y^\alpha) \tag{14}$$

The generalized Born rule postulates that the weights generate relative probabilities.

Important: The generalized Born rule depends on the dynamics, through the Ts. Unlike most of what we've discussed heretofore, the Born rule is *not* simply a "kinematic" property of Hilbert space operators alone.

The use of the generalized Born rule will become clearer through examples. For now let's just consider a couple of very simple ones. For two times, we can define a sample space of histories

$$\begin{aligned} Y^\alpha &= P_\alpha \otimes [\psi] \\ Y^0 &= 1 \otimes (1 - [\psi]) \end{aligned} \tag{15}$$

If we know that our system has property $[\psi]$ at time t_i, and are interested in the probability that it has property P_α at t_f, then we our

[3]You may notice a family resemblance between this concept and the "Wilson lines" of gauge theory. That might be an interesting thought to carry further.

recipe for the relative probability of histories takes us back to the special Born rule. On the other hand, we may only know that our system has the broader property Q at time t_i. Then it is appropriate to use the sample space

$$\begin{aligned} Y^\alpha &= P_\alpha \otimes Q \\ Y^0 &= 1 \otimes (1 - Q) \end{aligned} \tag{16}$$

and our recipe gives us the probabilities

$$\Pr(\alpha) = \frac{\operatorname{Tr} T(t_f, t_i)^\dagger P T(t_f, t_i) Q}{\operatorname{Tr} Q} \tag{17}$$

6. *Consistent Histories*

In order that the weight generated by the generalized Born behaves as a weight should, a non-trivial consistency condition must be satisfied. It is a gratifying and important meta-result, which in some sense justifies the consistent histories approach, that this consistency condition has a simple heuristic content, and that with its use one can defuse several quantum "paradoxes". (For examples of the latter, see the later chapters of Griffiths.)

The condition is this: We want that the weight of a union of histories should be the sum of the weights of the histories. We can incorporate unions by considering sums of the form

$$Y_\pi = \sum_\alpha \pi_\alpha Y^\alpha \tag{18}$$

where the π_α are 0 or 1. We would like for the weight of Y_π to be

$$W(Y_\pi) \stackrel{?}{=} \sum \pi_\alpha W(Y^\alpha) \tag{19}$$

On the other hand, a very simple calculation gives us

$$W(Y_\pi) = \sum_\beta \sum_\alpha \pi_\beta \pi_\alpha \operatorname{Tr} K(Y_\beta)^\dagger K(Y_\alpha) \tag{20}$$

We can insure that Eqn. (20) does give us Eqn. (19) if we have the *consistent histories condition*

$$\operatorname{Tr} K(Y_\beta)^\dagger K(Y_\alpha) = 0 \quad \text{for } \alpha \neq \beta \tag{21}$$

(In fact the weaker condition, that the left-hand side be antisymmetric in α, β, or equivalently that it is pure imaginary, is sufficient. The significance of this option, if any, is unclear.)

7. *Inconsistent History Example*

Sample spaces of histories based on two times always obey the consistent histories condition. (Checking this is a good exercise. Hint: It's trivial.)

For three times it is not so. We encounter an inconsistent set of histories already for a free spin-$\frac{1}{2}$ – i.e., a two-dimensional Hilbert space, and all the $T = 1$. Consider the sample space

$$\begin{aligned} Y^0 &= [z^-] \otimes 1 \otimes 1 \\ Y^1 &= [z^+] \otimes [x^+] \otimes [z^+] \\ Y^2 &= [z^+] \otimes [x^+] \otimes [z^-] \\ Y^3 &= [z^+] \otimes [x^-] \otimes [z^+] \\ Y^4 &= [z^+] \otimes [x^-] \otimes [z^-] \end{aligned} \tag{22}$$

where

$$[z^+] = \frac{1+\sigma_3}{2} \tag{23}$$

projects on spin up in the $\hat{z}$ direction, and so forth. One finds

$$\begin{aligned} \mathrm{Tr}\, K(Y^1)^\dagger K(Y^3) &= \mathrm{Tr}\frac{1+\sigma_3}{2}\frac{1+\sigma_1}{2}\frac{1+\sigma_3}{2}\frac{1-\sigma_1}{2}\frac{1+\sigma_3}{2} \\ &= \frac{1}{16}\mathrm{Tr}(1+\sigma_3+\sigma_1+i\sigma_2)(1+\sigma_3-\sigma_1-i\sigma_2) \\ &= \frac{1}{4} \neq 0 \end{aligned} \tag{24}$$

So this sample space fails the consistent histories condition. As it should! – for it corresponds to our spin, initially with spin up in the $\hat{z}$ direction, has both the property of having a definite value $\hat{x}$ direction, and then of having a definite value in the $\hat{z}$ direction. But – in the language of Copenhagen, translated from the Danish – the measurement of spin in the $\hat{x}$ direction destroys the possible property of having a definite spin in the $\hat{z}$ direction later, which we could have had, without the measurement.