Geometric Entropy: Evaluations

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Abstract

I review the calculation of geometric entropy in some simple cases, and comment on more general cases.

This is the second of three notes on geometric entropy. It covers, in detail, the evaluation of the entanglement entropy of an interval for 1+1 dimensional massless scalar field theory, and – by appeal to conformal symmetry and its anomaly – for conformal field theories in general. This is followed by some brief remarks about more general situations.

The presentation in this part, as regards 1+1 dimensional theories, is adapted from [1] and [2].

1 "Bare Hands" Evaluation For Massless Scalar

1. *Density Matrix as Path Integral*

   We shall evaluate the geometric entropy for a half-line, with appropriate cutoffs (as will appear), in massless 1+1 dimensional scalar field theory. Recall the ground state wave functional, whose arguments are real-space field configurations, can be expressed as a path integral

$$\Psi(L, R) \ \propto \ \int \mathcal{D}\phi \, e^{-A(\phi)} \tag{1}$$

   where[1]

$$A(\phi) \ = \ \frac{1}{2\pi} \int d\tau d\sigma \, \partial_\mu \phi \partial^\mu \phi \tag{2}$$

[1]For better or worse, we are following the conventions of [1].

is the Euclidean action, and the integral is over field configurations defined on the lower half-plane $\tau \leq 0$ that vanish at $\tau \to -\infty$, subject to

$$\phi(\sigma, \tau = 0^-) \;=\; \theta(-\sigma)L(\sigma) + \theta(\sigma)R(\sigma) \tag{3}$$

From this we express – still formally – our (unnormalized) density matrix as a similar path integral

$$\rho(R', R) \;=\; \int \mathcal{D}\phi \, e^{-A(\phi)} \tag{4}$$

but now with ϕ to be integrated over field configuration in the whole plane, vanishing at $\tau \to \pm\infty$, and subject to the discontinuous boundary conditions

$$\begin{aligned} \phi(\sigma_+, \tau = 0^+) &\;=\; R'(\sigma) \\ \phi(\sigma_+, \tau = 0^-) &\;=\; R(\sigma) \end{aligned} \tag{5}$$

on the positive half-line. For the replica trick, we will also want to use a similar expression for ρ^n, where the functional integral is taken over an n-sheeted cover of the plane, obtained by extending the range of the angular variable, with discontinuity of the form Eqn. (5) at the initial and final copies of the real axis.

2. *Step 1: Classical Action Rules*

   Since our functional integral involves a quadratic (Gaussian) positive-define action. For purposes of the functional integral, we can simply substitute the action of the *classical solution* with the given boundary conditions. Indeed, shifting the field by the classical solution, $A(\phi) \to A(\phi_{\text{cl.}} + \tilde{\phi})$, gives us the classical action term, a vanishing linear cross-term (since $\phi_{\text{cl.}}$ is a solution), and the universal factor $A(\tilde{\phi})$, which is independent of the boundary conditions.

3. *Steps 2: Stretched Coordinates and Poisson Kernel*

   It proves convenient, in this problem, to introduce complex variables $z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$, and to introduce the "stretched", logarithmic coordinate

$$z \;=\; e^{\eta} \tag{6}$$

(This will also serve us well when we pass to more general conformal field theories.) Then in the wave functional, where we integrate z over the lower half-plane, we shall have $\text{Im}\eta \leq 0$. In fact the whole lower z

half-plane is mapped into the strip $0 \leq \mathrm{Im}\eta \leq -\pi$. The right half-line $\sigma > 0, \tau = 0$ is mapped onto the top of the strip (i.e., the real axis), and the left half-line $\sigma < 0, \tau = 0$ is mapped onto the bottom of the strip.

To find the classical solutions, we call on our experience with electrostatic potential theory. We are solving Laplace's equation on a half-space, the specified boundary values – a classic problem, solved using the Poisson kernel. So we can write the answer directly:

$$\phi_{\mathrm{cl.}}(z,\bar{z}) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dw \left(\frac{1}{w-z} - \frac{1}{w-\bar{z}}\right)\Big(\theta(-w)L(w) + \theta(w)R(w)\Big) \tag{7}$$

where x runs over the real axis. (That this is the answer, can also be verified easily using standard results in complex variable theory.)

4. *Steps 3: Modes and Their Correlations*

Now we express L, R in terms of their Fourier modes *in the stretch variable*, according to

$$\begin{aligned} R(x) &= \int \frac{d\omega}{\sqrt{4\pi|\omega|}}\, e^{-i\omega x} r_\omega \\ L(x) &= \int \frac{d\omega}{\sqrt{4\pi|\omega|}}\, e^{-i\omega x} l_\omega \end{aligned} \tag{8}$$

where

$$w = (\mathrm{sign\,w})\, e^{x} \tag{9}$$

We have the reality conditions $r_{-\omega} = \bar{r_\omega}, l_{-\omega} = \bar{l_\omega}$. Inserting this into the Poisson kernel Eqn. (7), we find

$$\begin{aligned} \phi(\eta) \quad = \quad & \int \frac{d\omega}{\sqrt{4\pi|\omega|}} \times \qquad (10) \\ & e^{-i\omega\eta}\frac{1}{2\sinh\pi\omega}(e^{\pi\omega}r_\omega - l_\omega) - e^{-i\omega\bar{\eta}}\frac{1}{2\sinh\pi\omega}(e^{-\pi\omega}r_\omega - l_\omega) \end{aligned}$$

Again, one can verify directly that this is the required classical solution – it is a holomorphic function[2] with the required boundary values.

Note that the action integral, being of the form

$$\int dz d\bar{z}\, \partial\phi\bar{\partial}\phi \tag{11}$$

[2]To be more precise: It is a real sum of holomorphic and anti-holomorphic pieces.

can be evaluated directly in the transformed coordinates – the Jacobeans for differentials and derivatives cancel.

Now the "miracle" of our stretched coordinates is that the action integral is diagonal in 2×2 blocks, corresponding to $\pm\omega$. This is a vast simplification, which we would not have in truncated Fourier modes, which involve all frequencies. The Fourier modes in the stretched variables correspond to powers in the original variables. Unruh introduced their use in this sort of half-space problem. Using the integral

$$\int \partial e^{-i\omega'\eta}\bar{\partial}e^{-i\omega\bar{\eta}} \;=\; 2\pi\,\delta(\omega+\omega')\,\omega e^{\pi\omega}\,\sinh\pi\omega \tag{12}$$

we find

$$A_{\rm cl.} \;=\; \frac{1}{2}\int\frac{d\omega}{2\pi}\Big(\frac{\cosh\pi\omega}{\sinh\pi\omega}\,(\bar{r_\omega}r_\omega+\bar{l_\omega}l_\omega)-\frac{1}{\sinh\pi\omega}(\bar{r_\omega}l_\omega+\bar{l_\omega}r_\omega)\Big) \tag{13}$$

5. *Steps 4: Gaussian Integrals and the Density Matrix*

Given the action Eqn. (13), we have evaluated the wave function as

$$\Psi(\phi)\;\propto\;e^{-A_{\rm cl.}(\phi)} \tag{14}$$

For our purposes, it is most pleasant that we can thereby get the density matrix, summing over the left half-line variables, and its powers by doing Gaussian integrals over the mode variables l_ω!

Carrying through the integrals, we find, with a convenient normalization

$$\begin{aligned}\rho(R',R) \;&=\; \prod_{\omega>0}\frac{\sinh\pi\omega}{\cosh\pi\omega}\times \\ &\quad \exp\frac{-1}{2\sinh 2\pi\omega}\Big((\cosh 2\pi\omega(|r_\omega|^2+|r'_\omega|^2)-(\bar{r_\omega}r'_\omega+\bar{r_\omega}'r_\omega)\Big)\end{aligned} \tag{15}$$

One can then compute higher powers by doing additional Gaussian integrals. By induction, one proves

$$\begin{aligned}{\rm Tr}\,\rho^n \;&=\; \frac{(2\sinh\pi\omega)^{2n}}{(2\sinh n\pi\omega)^2} \\ \frac{\rho^n}{{\rm Tr}\,\rho^n} \;&=\; \prod_{\omega>0}\frac{\sinh n\pi\omega}{\cosh n\pi\omega}\times \\ &\quad \exp\frac{-1}{2\sinh 2n\pi\omega}\Big((\cosh 2n\pi\omega(|r_\omega|^2+|r'_\omega|^2)-(\bar{r_\omega}r'_\omega+\bar{r_\omega}'r_\omega)\Big)\end{aligned} \tag{16}$$

6. *Step 5: Geometric Entropy Result*

   Now we are ready to apply the replica trick, to get the geometric entropy, according to

$$S(\rho) \;=\; (1 - n\frac{d}{dn})_{n\to 1} \ln \operatorname{Tr} \rho^n \tag{17}$$

   acting on the traces in Eqn. (16). In fact we can throw away the numerator, since its logarithm is linear in n, and its contribution cancels. So we have, after a bit of algebra and an integration by parts,

$$S(\rho) \;\sim\; \int_0^\infty \frac{d\omega}{?} \, \frac{8\pi\omega}{e^{2\pi\omega} - 1} \tag{18}$$

   To give a value to the question mark, and replace the $\sim$ with $=$, we must put some grit into our formal procedures. If we regulate in a box of length L, the density of modes will correspond to the measure $\frac{d\omega}{2\pi}$, and the mode-count L times that. So we find

$$S(\rho) \;=\; L \int_0^\infty d\omega \, \frac{8\pi\omega}{e^{2\pi\omega} - 1} \;=\; \frac{1}{6} L \tag{19}$$

   In evaluating the integral, one expands the denominator in a power series, and encounters Euler's famous sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6} \tag{20}$$

   Now we should remember that L is the length *in the stretched coordinates*, since we've done the calculation using those coordinates. In terms of the original coordinates, we have

$$S(\rho) \;=\; \frac{1}{6} \ln \frac{\Sigma}{\epsilon} \tag{21}$$

   where Σ and ϵ are infrared and ultraviolet cutoffs.

7. *What Does It Mean?*

   At first encounter it might seem quite an anticlimax, after so much work, to get an answer that depends, in its entirety, on the cutoffs we impose! But on reflection, we realize that this result is both appropriate and meaningful:

- The ultraviolet cutoff reflects the high degree of entanglement between nearby points, which in turn reflects the fact that the ground state is constructed in a compromise between the cost of field gradients and the desirability of letting the wave functional spread (heuristically: spontaneous vacuum fluctuations). The fluctuations on small scales will be highly constrained, or correlated, and a sharp division into L and R will expose those correlations. In any case, the ultraviolet cutoff has a simple universal character, and as usual we can strive to define physically meaningful quantities where that cutoff dependence cancels.
- The infrared cutoff is naturally associated with the volume (i.e., length) of our system. If we compare systems of different sizes, and apply a common ultraviolet cutoff, then the difference between their geometric entropies is perfectly well-defined and finite, in the form

$$S(\rho_1) - S(\rho_2) \;=\; \frac{1}{6} \ln \frac{\Sigma_1}{\Sigma_2} \tag{22}$$

- Our massless boson theory is scale invariant, and so we cannot expect to find an absolute, size-dependent result for the geometric entropy of any particular system. But *relative* sizes are meaningful, and we might hope to find dependence on size ratios. Constraint and hope are here reconciled, through the properties of logarithms (with cutoff!). This is typical of how we get such "absolutely meaningless, but relatively sensible" results in physics – broadly similar shenanigans occur in BCS theory and in the foundations of QCD, for example.

Now we will vastly generalize the preceding result, by exploiting the techniques of 1+1 dimensional conformal field theory. This is a highly developed subject with a lovely and extensive, but intricate and specialized, body of technique. Our use of conformal field theory technology will not be not tremendously demanding, by the standards of the field, but it would require a long digression to develop even what we use from scratch, and I don't want to pause for that[3] My compromise will be to try to isolate a few specific points where serious machinery is brought to bear, and state the necessary results clearly, though without proof. There are several attractive presentations of conformal field theory basics; [3] is a nice short entry-level one.

[3]Nor to labor over it.

1. *Orientation in Conformal Field Theory*

Conformal transformations are transformations that implement position-dependent changes in length scale, but leave angles invariant. In terms of a metric, we have

$$g_{\mu\nu}(x) \;\rightarrow\; e^{\sigma}(x) g_{\mu\nu}(x) \tag{23}$$

– the so-called Weyl symmetry. Note that in this transformation we do *not* transform coordinates. If we want to stay within the realm of flat-space theories, we can look for pure coordinate changes (differomorphisms) that rescale the metric, but then refuse to transform the metric! That implements a sort of inverse Weyl transformation: a diffeomorphism, followed by "undoing" the associated metric transformation through a Weyl transformation. These are the sort of conformal transformations that will primarily concern us here. Conformal transformations are, evidently, a generalization of scale transformations. In 1+1 dimensions, or in Euclidean 2 dimensions, the conformal transformations make an infinite dimensional group. Indeed, it is a famous, elementary result in the theory of complex variables that holomorphic, or anti-holomorphic, mappings define conformal transformations. In higher dimensions the conformal transformations are much more restricted. Here I will stick to 2 dimensions.

On the physics side, there are large classes of Lagrangian field theories that are conformal invariant at the classical level. It's easy to see this in the form of Weyl symmetry: For example, we can have several scalar fields ϕ^j, and the action

$$A \;=\; \int d^2x\, \sqrt{g}\, g^{\alpha\beta} F(\phi)_{jk}\, \partial_\alpha \phi^j \partial_\beta \phi^k \tag{24}$$

where F is an arbitrary (positive) function. At the quantum level things are not quite so simple, but there are still many known constructions that lead to conformal invariant field theories.

Conformal symmetry is different from more familiar symmetries in that the Hamiltonian is covariant, rather than invariant. Thus we do not get simply degenerate multiplets, but rather elaborate spectra from representing conformal symmetry.

Conformal symmetry is used in string theory, especially in connection with the world-sheet theory, in the construction of solvable models, in the theory of low-dimensional critical phenomena, and in some other chapters of condensed matter theory.

The algebra associated with conformal symmetry has infinitesimal generators L_m corresponding to the infinitesimal versions of the complex mappings $z \to z^m$. They satisfy the *Virasoro algebra*

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{25}$$

The first term on the right-hand side has a simple geometrical origin, but the second term requires comment. It is a sort of anomaly. Once we exponentiate the operators, it is a pure phase term, so it renders the representation of conformal symmetry into a *projective* representation – as with the spin-$\frac{1}{2}$ representation of rotations, one cannot fix the phases while remaining consistent with the group structure. c is a number, the *central charge*. Its value varies from one conformal field theory to another.

The central charge appears in several other contexts. It is the coefficient in the singularity that arises when two insertions of the energy-momentum tensor approach one another, and it appears in the transformation law for the energy-momentum tensor under conformal mappings, where it governs an "anomalous" correction to normal tensor behavior. That feature will play a key role below.

Another feature of conformal field theories that we will exploit is *modular invariance*. It arises when one considers the behavior of the theory on a torus T^2, regarded as a one-dimensional complex manifold. Global conformal transformations can change tori of different shapes into one another, but not in entirely arbitrary ways. The ratio τ of the lengths of real and imaginary cycles (with respect to some complex parameterization) is called the modulus of the torus. Different choices of complex parameters can lead to different modular parameters, according to the transformation law

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{26}$$

with a, b, c, d integers and $ad - bc = 1$. In particular, with the choice $a = d = 0; b = -c = 1$, we can interchange the sizes of the two cycles. That possibility will be very important to us below.

2. *Geometric Entropy as an Invariant*

In the quantum theory, global conformal transformations maintain the topology of inside and out and leave the vacuum invariant. Therefore the geometric entropy, relative to vaccum, associated with a given region is invariant under such transformations!

3. *Mapping to Thermality*

I will now revisit the calculation of geometric entropy for an interval in massless scalar field theory – which is, of course, a conformal field theory – using different techniques, emphasizing the conformal mapping. One of these techniques generalizes easily to general conformal field theories.

We take as our universe the interval $[0, \Lambda)$ with periodic boundary conditions. We take our subsystems to be based on $\mathcal{R}_1 = [\epsilon_1, \Sigma - \epsilon_2)$ and $\mathcal{R}_2 = [\Sigma + \epsilon_2, \Lambda - \epsilon_1)$, thus introducing ultraviolet regulators ϵ_1, ϵ_2. Their implementation will emerge in the course of the calculation. We will denote the spatial variable by σ, and extend it to the complex variable $\zeta = \sigma + i\tau$ for a region of negative τ.

Our first conformal mapping,

$$w \;=\; -\frac{\sin\frac{\pi}{\Lambda}(\zeta - \Sigma)}{\sin\frac{\pi}{\Lambda}\zeta} \tag{27}$$

maps $\mathcal{R}_1$ into the interval

$$\mathcal{R}_1 \;=\; [\frac{\sin\frac{\pi}{\Lambda}\Sigma}{\sin\frac{\pi}{\Lambda}\epsilon_1}, \frac{\sin\frac{\pi}{\Lambda}\epsilon_2}{\sin\frac{\pi}{\Lambda}\Sigma}) \tag{28}$$

on the positive real axis and $\mathcal{R}_2$ to the symmetrical interval on the negative real axis. We are aiming to project on to the ground state with an appropriate functional integral. We can do this by taking our standard

$$\int \mathcal{D}\phi\, e^{-A(\phi)}$$

over the annular region we sweep out as we rotate $\mathcal{R}_1$ into $\mathcal{R}_2$ by counterclockwise around the origin as axis, imposing vanishing boundary conditions on the bottom of that big annulus. This, operationally, is how we implement our regulators. The details of that implementation should not matter, once we extract cutoff-independent (universal) properties.

Now we make a second conformal mapping

$$z \;=\; \frac{\ln w}{k} \tag{29}$$

where k is a redundant "check" parameter, which has better cancel from our final result. After the mapping, our annulus becomes a rectangle, bounded by the image of $\mathcal{R}_1$ on the real axis, a parallel image of $\mathcal{R}_2$ on the line $\mathrm{Im}\, z = -\pi$, and sides of length $\frac{\pi}{k}$. The common length of $\mathcal{R}_1, \mathcal{R}_2$ is

$$L = \frac{2}{\kappa} \ln\left(\frac{\Lambda}{\pi\epsilon} \sin\frac{\pi\Sigma}{\Lambda}\right) \tag{30}$$

where $\epsilon = \sqrt{\epsilon_1\epsilon_2}$.

To get the traces we require for the replica trick, we need to extend region downward by a factor n, identify the top and bottom, and sum. But now we recognize that this is exactly the procedure we'd use to calculate the thermodynamic partition function, at inverse temperature

$$\beta = \frac{2\pi n}{k} \tag{31}$$

in a box of length L, where we've imposed periodic boundary conditions. Our formula for completing the calculation of the geometric entropy, through the replica trick, by going from the traces to geometric entropy, is identical to the statistical mechanic recipe for calculating the thermodynamic entropy of the gas from the partition function, after identifying $n \propto \beta$, as we've reviewed previously. Since we've already done a very similar calculation, and the statistical mechanics problem is standard in any case, I'll just quote the answer

$$S = \frac{1}{3} \ln\left(\frac{\Lambda}{\pi\epsilon} \sin\frac{\pi\Sigma}{\Lambda}\right) \propto LT = \frac{L}{\beta(n=1)} \tag{32}$$

(We have a factor of two here, compared to the previous result, because our intervals now have two distinct ends.)

There is another way of looking at the final stages of this calculation, which we will exploit momentarily. The partition function of our massless scalar field on a torus with sides L, $\frac{2\pi n}{k}$ can be written in the form that emerges naturally from conformal field theory

$$\begin{aligned} Z &= \frac{1}{\eta\bar{\eta}} \\ \eta &= q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \end{aligned} \tag{33}$$

where

$$q = e^{2\pi i\tau}$$

$$\tau = \frac{2\pi i n}{kL} \tag{34}$$

τ, the ratio of sides, is the modular parameter. This may look unfamiliar at first sight, but after spelling out

$$1 - q^k = 1 - e^{-\frac{4\pi^2 nr}{kL}} \tag{35}$$

we recognize the product as arising from the contributions of the various momentum modes to the partition function, where the exponential encodes the Boltzmann factor, and we sum over the possible occupancies. We have factors in η and in $\bar{\eta}$ for the left-moving and right-moving modes. The prefactor $q^{\frac{1}{24}}$ renders the partition function modular invariant. It has no effect on our calculation of geometric entropy, since its logarithm is linear in n, and cancels in the operation $1 - n\frac{d}{dn}$.

Now if carried through our mappings in terms of the abstract Hamiltonians $L_0, \bar{L}_0$, available in the general conformal field theory, and kept track of its transformations we would arrive through similar steps at the recognizable generalization

$$Z = q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}} \operatorname{tr} q^{L_0}\bar{q}^{\bar{L}_0} \tag{36}$$

where now we allow different central charges, as well as different Hamiltonians, for the two directions of propagation. In the free boson case we have (famously) $c = \bar{c} = 1$.

4. *Modular Transformation*

If we make a modular transformation, in effect interchanging the sides of the torus, we change $n \to \frac{1}{n}$. Also, importantly, we exchange the large box size L for a very low temperature! Thus the contributions of the excitations go away, and

$$\operatorname{tr} q^{L_0}\bar{q}^{\bar{L}_0} \to 1 \tag{37}$$

The whole answer comes from the funny prefactors – which no longer cancel, due to the change in sign of $n\frac{d}{dn}$! In this way, we derive the general answer

$$S = \frac{c + \bar{c}}{6} \ln\left(\frac{\Lambda}{\pi\epsilon} \sin\frac{\pi\Sigma}{\Lambda}\right) \tag{38}$$

The replacement of complicated high-energy mode sums by contributions of a few (or one) state is a "typical miracle" in the application of anomalous symmetries.

5. What Does It Mean?

   By invoking more powerful machinery, we have not only re-derived our earlier result for geometric entropy in massless boson theory in a more conceptual way, but also vastly generalized it. Our earlier interpretive remarks, in the preceding section, remain pertinent in the general case, and there are a few additional points of note:

   - geometric entropy generally very difficult to compute, and there are few analytical results, even for free field theory. It is remarkable, therefore, that we find a simple, rigorous result applicable to a broad class of theories, including some very complicated ones, with highly non-trivial interactions.
   - reading it the other way, the geometric entropy supplies a new characterization of the central charge, whose physical meaning is direct and easily stated
   - experiments? numerical; diagnostic

2 Heat Kernel and Higher Dimensions

Another technique we can apply, in conjunction with the replica trick, to calculate some aspects of geometric entropy is the so-called heat kernel method, first used in this context in [4]. It is especially powerful as regards the contributions from high-frequency modes, and can be used in any dimension.

Here I will illustrate how the method works by reference to the same circle of problems we've been considering. We consider, in particular, Euclidean field theory on the space $C_\delta \times M_{D-2}$, where C_δ is a two dimensional cone of radius L and deficit angle δ and M_{D-2} is a flat $D-2$ dimensional transverse space with total volume V_{D-2}. We will consider, specifically, the theory of a free massive scalar field of this space, with the goal of obtaining the geometric entropy of a half-line times M_{D-2}, tracing over a complementary half-line times M_{D-2}. The relevant "partition function" path integral is Gaussian, and the result can be expressed as a functional determinant

$$\ln Z_\delta \;=\; -\frac{1}{2}\ln\det(-\Delta+\mu^2) \tag{39}$$

where Δ is the Laplacian and μ the mass. The heat kernel is defined in terms of the eigenvalues $-\lambda_n$ of the Laplacian as

$$\zeta(t) \;=\; \operatorname{tr} e^{t\Delta} \;=\; \sum_n e^{-\lambda_n t} \tag{40}$$

The reason for the name, "heat kernel", deserves brief comment. Clearly we have

$$(\frac{\partial}{\partial t} - \Delta)e^{t\Delta} \ = \ 0 \tag{41}$$

so $e^{t\Delta}$, acting on a function $f(x,0)$, produces a solution $f(x,t)$ of the heat equation. If we express $f(x,0) = \delta(x)$, and write the Dirac $\delta(x)$ as the sum

$$\delta(x) \ = \ \sum_n \phi_n(x)\phi_n(x) \tag{42}$$

over a complete set of eigenvalues, we see that the heat kernel describes the evolution starting from a δ function source. At the risk of belaboring the obvious, let me also comment that the heat equation is the imaginary time version of the Schrödinger equation, so that we are dealing with a close relative of the unitary evolution operator e^{-iHt}.

Using the heat kernel, we can define a regulated version of our functional determinant through

$$\ln\det(-\Delta + \mu^2) \ = \ -\int_{\epsilon^2}^{\infty} \frac{dt}{t}\zeta(t)e^{-\mu^2 t} \tag{43}$$

Let me add a few words of explanation to this construction. Its basic unit is the integral

$$I(u) \ \stackrel{??}{=} \ \int_0^{\infty} \frac{dt}{t}\, e^{-tu} \tag{44}$$

This integral is divergent at the $t = 0$ end. Before facing up to that embarrassment, let's note that the formal derivative

$$\frac{d}{du}\, I(u) \ = \ \int_0^{\infty} \frac{dt}{}e^{-tu} \ = \ \frac{1}{u} \tag{45}$$

which suggests that in some sense $I(u) \ = \ \ln u$, which is what we want. To make this more legitimate and precise, let us consider the well-defined differences

$$I(u;\mu^2) - I(u;\nu^2) \ = \ \int_{\epsilon^2}^{\infty} \frac{dt}{t}\zeta(t)e^{-\mu^2 t} \ - \ \int_{\epsilon^2}^{\infty} \frac{dt}{t}\zeta(t)e^{-\nu^2 t} \tag{46}$$

Then we have, by a kosher version of the same mathematics,

$$I(u;\mu^2) - I(u;\nu^2) \ = \ \ln(u + \mu^2) - \ln(u + \nu^2) \tag{47}$$

Now on physical grounds we expect that as the mass $\nu \to \infty$, we are entitled to ignore the very massive field – and that is what we do! This kind of procedure, in the context of quantum field theory and its renormalization, is called Pauli-Villars regulation. In that context one can often, by careful attention to the limiting process, extract finite, "renormalized" values for physical quantities as the fictitious regulator fields are removed. It is an interesting challenge, to identify quantities of this type related to geometric entropy. (This would supply, I think, a rigorous version of the hand-waving discussion of "universality" above.)

Having motivated the formal maneuver in Eqn.(43) – with $\epsilon = 0$! – we then introduce a different regulator, namely ϵ, that has a similar qualitative effect, and removes the need for a separate, explicit Pauli-Villars procedure. Indeed, by excluding small t in the integral, we effectively kill the contribution of very large eigenvalues (and, for large masses, they're all large).

There are powerful mathematical results relating the $t \to 0$ behavior of heat kernels to the geometry of the underlying manifold. These results make precise, and generalize, the "modes per unit volume" arguments physicists invoke in many contexts, such as the treatment of black-body radiation [5]. On the two-dimension manifold C_δ one has, for example

$$\zeta_2(t) \;=\; t^{-1}\frac{2\pi-\delta}{8\pi}L^2 \;+\; \frac{1}{12}(\frac{2\pi}{\delta} - \frac{\delta}{2\pi}) \;+\; O(t/L^2) \tag{48}$$

One the full manifold $C_\delta \times M_{D-2}$ there is an extra factor

$$\frac{V_{D-2}}{(4\pi t)^{\frac{D-2}{2}}} \tag{49}$$

from the trivial Laplacian on the transverse space. (In that case, one is literally solving the heat equation! Assembling the pieces, we have

$$S \;=\; \frac{V_{D-2}}{(4\pi)^{\frac{D-2}{2}}} \int\limits_{\epsilon^2}^{\infty} \frac{dt}{t^{\frac{D}{2}}}\, e^{-\mu^2 t}\,(\frac{1}{12} \;+\; O(t/L^2)) \tag{50}$$

This leads, for $D = 2$, back to our earlier result for the ultraviolet divergence, with the mass appearing only in subleading terms

$$S_2 \;=\; \frac{1}{6}\ln\frac{1}{\epsilon} \;+\; ... \tag{51}$$

while in higher dimensions the transverse space modifies this to

$$S_D \;=\; \frac{V_{D-2}}{(2\sqrt{\pi})^{D-2}}\,(\frac{D}{2} - 1)\,\epsilon^{2-D} \tag{52}$$

In the black hole context, V_{D-2} is the area of the horizon.

For $D > 2$ the numerical coefficient is non-universal, because it depends on the scale of the cutoff. One can hope to extract universal quantities from sub-leading terms.

In the black hole case, we do get a leading contribution proportional to the area. If we choose the cutoff to be of order the Planck scale, we can imagine getting some or all of the standard black hole entropy, formally, from entanglement. At present, however, no justification for such procedures is on the horizon.

References

[1] F. Larsen, F. Wilczek *Annals of Physics* **243** 280 (1995).

[2] C. Holzhey, F. Larsen and F. Wilczek *Nucl. Phys.* **B424** 443 (1994).

[3] D. Tong http://bit.ly/1e4WJHM

[4] C. Callan and F. Wilczek *Phys. Lett.* **B333** 55 (1994).

[5] R. Balian, C. Bloch *Annals of Physics* **64** 271 (1971); O. Alvarez *Nucl. Phys.* **B216** 125 (1983).