Entangled Histories

Concepts and Examples
The consistent histories approach to quantum mechanics, pioneered by Robert Griffiths, gives a mathematically and logically coherent formulation of the Copenhagen interpretation*.
Its results do not differ from those given by standard procedures, in cases where their application is clear.

But it clarifies - and if accepted, removes - several “paradoxes”, where the correct use of quantum mechanics does not seem clear to everyone.
Several major physicists, notably including Gell Mann and Hartle, have found the consistent histories approach attractive. (For G-M & H, quantum cosmology is a strong motivation.)
The consistent histories approach offers a useful framework for discussing subtle “history-dependent” phenomena in quantum mechanics, notably including anyon/Majorino physics, quantum computing in general, and entangled radiation fields.

It also brings out questions and possibilities that otherwise might be hard to perceive.
Consistent Histories
Here I will consider only non-relativistic quantum mechanics.

We represent properties by projection operators $P_\alpha$, and physical variables by complete sets of alternative properties:
Geometrically: Decomposition of Hilbert space into orthogonal subspaces.
A state $|\psi\rangle$ has property $P_{\alpha}$ if $P_{\alpha} |\psi\rangle = |\psi\rangle$, and it has property NOT $P_{\alpha}$ if $P_{\alpha} |\psi\rangle = 0$.

Otherwise, the property is *undefined* on that state.

Non-commuting properties generally can’t both be defined at once.
We’d like to extend those concepts from states to histories.

Choose a set of times $t_f > t_n > t_{n-1} > ... > t_1 > t_i$ and define the history Hilbert space
\mathcal{H} = H^{t_f} \otimes \ldots \otimes H^{t_1} \otimes H^{t_i}
The standard definition - which we’ll modify later - of a property in this history space is simply a product of properties at different times:
\[ Y = P^{t_f} \otimes P^{t_n} \ldots \otimes P^{t_1} \otimes P^{t_i} \]

Interpretation: State has property \( P^{t_j} \) at time \( t_j \).
We introduce dynamics through unitary evolution or bridging operators $T(t_b, t_a)$, that map $H^{t_a}$ to $H^{t_b}$ and satisfy
\[
\begin{align*}
T(t_b, t_a)^{-1} &= T(t_b, t_a)^\dagger = T(t_a, t_b) \\
T(t_c, t_b)T(t_b, t_a) &= T(t_c, t_a) \\
T(t_a, t_a) &= 1
\end{align*}
\]
Now we want to define probabilities for properties of histories.

First, let’s cast the usual Born rule in a convenient form:
\[ Pr = \left\| P^{t_1} T(t_1, t_0) |\psi\rangle \right\|^2 \]

\[ Pr = \langle \psi | T(t_1, t_0)^\dagger P^{t_1} T(t_1, t_0) |\psi\rangle \]
\[ = \text{Tr} \left( T(t_1, t_0)^\dagger P^{t_1} T(t_1, t_0) [\psi]^{t_0} \right) \]

\[ [\psi]^{t_0} \equiv |\psi\rangle \langle \psi| \]

projection operator at \( t_0 \)
We can now generalize to properties of histories:
probability or “weight”

\[ W(Y^\alpha) = \text{Tr} \ K(Y^\alpha)^\dagger K(Y^\alpha) \]

\[ K(Y^\alpha) \equiv P_{\alpha}^{t_f} T(t_f, t_{f-1}) P_{\alpha}^{t_f-1} \ldots T(t_2, t_1) P_{\alpha}^{t_1} T(t_1, t_i) P_{\alpha}^{t_i} \]

In the K operation, we replace \( \otimes \) by T.
Now we’re ready to discuss physical variables based on histories.

As in the case of states, the idea is to have a classification by properties:
Geometrically: Decomposition of Hilbert space into orthogonal subspaces.

\[ 1 = \sum_{\alpha} Y_{\alpha} \]
\[ Y_{\alpha}^\dagger = Y_{\alpha} \]
\[ Y_{\alpha} Y_{\beta} = \delta_{\alpha\beta} Y_{\alpha} \]
We want to use the generalized Born rule to assign probabilities to different values of such quantum variables, or in other words to (some) subspaces of history Hilbert space.

Here a new phenomenon arises. If we want the proposed probabilities to behave as probabilities - that is, to be additive - there is an additional, non-trivial condition:
\[ Y_\pi = \sum_\alpha \pi_\alpha Y^\alpha \]

\[ \pi_\alpha = 0 \text{ or } 1 \]

\[ W(Y_\pi) = \sum \pi_\alpha W(Y^\alpha) \]

\[
\text{Re } \text{Tr } K(Y_\beta)^\dagger K(Y_\alpha) = 0 \quad \text{for } \alpha \neq \beta
\]

\[ \text{weak vs. strong} \quad \text{consistent histories condition} \]
The consistent histories condition is a constraint on the kinds of quantum questions we can handle using classical logic.

It is similar in spirit to the constraint on using noncommuting observables, but it has independent significance.

Most if not all of the famous “paradoxes” of quantum theory arise from violating these constraints.
History Observables
So far we have only considered properties (projectors) that factorize into a product of projections at different times. It is natural to consider, as potential observables, more general Hermitean operators in history Hilbert space.
In order for the potential “observations” to make logical sense, we require that the projections onto the eigenspaces must obey the consistent histories condition, and must commute.

That can happen due to cancellations among non-trivial cross terms that arise in the factorized basis.

In that way, we get a proper expansion of the notion of history observables. It reflects possibilities for entanglement across time.
Toy example, for spin 1/2, trivial dynamics (T=1) :

\[ A = \sigma_2 \otimes \sigma_1 \]
\[ B = \sigma_1 \otimes \sigma_3 \]

\[ K(A) = -i\sigma_3 \]
\[ K(B) = -i\sigma_2 \]

consistent!
\[ v_{++} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + i \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - i \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \]

For \[ [\psi] = \frac{1 + \hat{n} \cdot \vec{\sigma}}{2} \] Schrödinger history \[ [\psi] \otimes [\psi] \equiv \Psi \]

\[
\begin{align*}
\text{Tr } K(++)^\dagger \Psi &= \frac{1}{4} \left( in_3 + in_2 - in_1 \right) \rightarrow \frac{1}{4} \left( n_3 + n_2 - n_1 \right)^2 \\
\text{Tr } K(+-)^\dagger \Psi &= \frac{1}{4} \left( in_3 - in_2 + in_1 \right) \rightarrow \frac{1}{4} \left( n_3 - n_2 + n_1 \right)^2 \\
\text{Tr } K(--)^\dagger \Psi &= \frac{1}{4} \left( -in_3 + in_2 + in_1 \right) \rightarrow \frac{1}{4} \left( -n_3 + n_2 + n_1 \right)^2 \\
\text{Tr } K(--)^\dagger \Psi &= \frac{1}{4} \left( -in_3 - in_2 - in_1 \right) \rightarrow \frac{1}{4} \left( -n_3 - n_2 - n_1 \right)^2
\end{align*}
\]
Classic inconsistent history
variable:

\[ Y^0 = \left[ \frac{1 - \sigma_3}{2} \right] \otimes 1 \otimes 1 \]
\[ Y^1 = \left[ \frac{1 + \sigma_3}{2} \right] \otimes \left[ \frac{1 + \sigma_1}{2} \right] \otimes \left[ \frac{1 + \sigma_3}{2} \right] \]
\[ Y^2 = \left[ \frac{1 + \sigma_3}{2} \right] \otimes \left[ \frac{1 + \sigma_1}{2} \right] \otimes \left[ \frac{1 - \sigma_3}{2} \right] \]
\[ Y^3 = \left[ \frac{1 + \sigma_3}{2} \right] \otimes \left[ \frac{1 - \sigma_1}{2} \right] \otimes \left[ \frac{1 + \sigma_3}{2} \right] \]
\[ Y^4 = \left[ \frac{1 + \sigma_3}{2} \right] \otimes \left[ \frac{1 - \sigma_1}{2} \right] \otimes \left[ \frac{1 - \sigma_3}{2} \right] \]

\[ K(Y^0) = \frac{1 - \sigma_3}{2} \]
\[ K(Y^1) = \frac{1}{4} (1 + \sigma_3) \]
\[ K(Y^2) = \frac{1}{4} (\sigma_1 + i\sigma_2) \]
\[ K(Y^3) = \frac{1}{4} (1 + \sigma_3) \]
\[ K(Y^4) = -\frac{1}{4} (\sigma_1 + i\sigma_2) \]
\[
\text{Tr } K(Y^1)^\dagger K(Y^3) = \frac{1}{4} \neq 0
\]

But the coarser variable based on \( Y^0, Y^1 + Y^4, Y^2 + Y^3 \) is both non-trivial and consistent.
To get similar “algebraic” satisfaction of the consistent histories condition for an interacting system, we should use Heisenberg representation operators.

In terms of Heisenberg representation operators, the K operation essentially reduces to taking the (time-ordered) product.