Survey of Symmetry and Conservation Laws: More Poincare

Symmetry under space and time translations is the mathematical expression of the uniformity of physical laws. It says that the same laws hold everywhere and everywhen. These translations generate an abelian group, the additive group $R^1 \oplus R^3$.

The associated conservation laws are the conservation of momentum and of energy.

Now of course the world as a whole is *not* invariant under space and time translations – I'm here, and you're there, the universe is expanding, and we're all getting older. But we assume that we can ignore the effect of distant objects, or summarize their influence in a few parameters like (specifically) the values of local gravitational and electromagnetic fields, and get an accurate description, using the same laws anywhere and anywhen. There's nothing logically inevitable about that assumption, but it has served physics well.

We further assume, with the same sort of understanding, that the basic laws are invariant under rotations and boosts, together known as Lorentz transformations. These are the transformations that leave the proper distance between events invariant. They define the group $O(1, 3)$.

It is natural to consider space-time translations together with boosts and rotations, since they do *not* mutually commute. In general, we will break down our symmetries into mutually commutative units, which can be analyzed independently.

An important refinement emerges upon closer analysis. We are requiring invariance under transformations L such that

$$(Lx)^T \, \eta \, Lx = x^T \, \eta \, x \tag{1}$$

where the diagonal matrix

$$\eta \;\equiv\; \text{diagonal}\,(-1, 1, 1, 1) \tag{2}$$

implements proper distance. We can take L to be a linear transformation (why?), and since Eqn. (1) is supposed to hold for any x, we find that

$$L^T \, \eta \, L \;=\; \eta \tag{3}$$

Now, reflecting the 1+3 split between time and space, let us write L in block diagonal form, and consider separately rotations, of the form

$$L \;=\; \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \tag{4}$$

where R is a 3×3 matrix, and boosts which we'll analyze presently.

The invariance condition on R is the familiar orthogonality condition

$$R^T R \;=\; 1 \tag{5}$$

which implies for the determinant

$$\det R = \pm 1 \tag{6}$$

The transformations with $\det R = 1$ are proper rotations. They can be continuously related to the identity transformation (why?). The transformations with $\det R = -1$ are improper rotations. Obviously, they cannot be continuously related to the identity transformation. Spatial inversion P is the special transformation that simply reverses all three spatial coordinates: $R = -1$. P is an improper rotation that commutes with all proper rotations, and any improper transformation R can be written as P followed by the proper rotation $-R$.

A straightforward analysis of the invariance condition Eqn. (3) shows that boosts along the x direction can be written in the form

$$\begin{pmatrix} \epsilon_1 \cosh r & \sinh r & 0 & 0 \\ \epsilon_1\epsilon_2 \sinh r & \epsilon_2 \cosh r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

with $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. Again, the transformations continuously connected to the identity have $\epsilon_1 = \epsilon_2 = 1$. These are the proper boosts. To preserve the orientation of time we take $\epsilon_1 = 1$, and to preserve the orientation of space we take $\epsilon_2 = 1$. The special transformation T, corresponding to

$$L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

plays a similar role, for time, to that of P for space.

We can express any Lorentz transformation that is continuously connected to the identity as products proper rotations and proper boosts, in a way similar to Euler angles for rotations. That gives us the "proper, isochronous" Lorentz transformations, which preserve the *orientations* of space and time. They close under multiplication, giving us the proper isochronous Lorentz group. It is denoted $SO_+(1,3)$, where the "S", for Special, indicates determinant unity, and the + refers to preservation of the time orientation.

The most general Lorentz transformation takes precisely one of the forms L_0, L_0P, L_0T, L_0PT, where L_0 is a proper isochronous transformation. So it makes good mathematical sense to consider the proper, isochronous Lorentz transformations first, and then P and T sepraately. For physics, it is crucial to do so. We return to the spatial inversion or parity operation P and the time reversal operation T immediately below.

Combining time and space translations with the proper isochronous Lorentz transformations gives us the Poincaré group of transformations.

We will work under the hypothesis that the Poincaré transformations are exact symmetries of the basic laws of Nature. Violation of this hypothesis would be shocking, not only due to its empirical success to date, but also because it would challenge the formal consistency of general relativity, which is essentially the gauged or local version of Lorentz invariance.

Corresponding to the ten transformations of Poincaré symmetry, we have ten conservation laws. We've already mentioned energy and momentum, associated with the space-time translations. We also have three components of angular momentum, associated with rotations. The three conservation laws associated with boosts are for some reason less celebrated, and don't have a name of their own, but they are also fundamental. They are conservation of the three components of the velocity of the center of mass. (It is amazing to see how elementary mechanics textbooks usually "prove" this result by sneaking in different, *ad hoc* assumptions.)

Elementary particles should be associated with *irreducible* representations of symmetry. The irreducible representations of the space-time translations are states of definite energy and momentum. Under rotations and boosts, energy-momenta related by

$$E^2 - \vec{p}^2 \ = \ m^2 \tag{9}$$

for any fixed value of m^2, transform among themselves (they form "orbits" of the symmetry). Thus for example all the colors of the rainbow, which are light of different frequencies, in whatever direction the rainbow happens to be occur, are all regarded as manifestations of a single entity, namely the photon. If you have one, given Poincar*é* symmetry, you have them all.

If $m^2 > 0$ then we can boost to a frame in which $\vec{p} = 0$. The spatial rotations leave this frame invariant, so our elementary particle should also support an irreducible representation of spatial rotations. This leads to the spin quantum number.

Wigner showed that, conversely, given an $m^2 > 0$ and a value of the spin, there is a unique irreducible representation of Poincar*é* symmetry.

$m = 0$ is special, and different. Now we can't bring $(E, \vec{p}) \to (m, 0)$ by boosts, but rather $(E, \vec{p}) \to (1, 0, 0, 1)$. The Poincar*é* transformations that leave this fixed are isomorphic, not to rotations in three dimensions, but to a different 3-parameter group, the group $E(2)$ of Euclidean motions (translations and rotations) in two dimensions! This is easiest to show using infinitesimal versions of the Lorentz transformations [shown in class, supplementary note later]. Representations of this group with non-zero "momentum" correspond to a curious kind of continuous spin, that so far has not played a significant role in physics. Restricting to the (now trivial, one-dimensional) rotation subgroup, we have a kind of spin *around one axis*, namely the $\hat{z}$ axis. We do not need to fill out a multiplet of spins, but can take just one fixed value. This corresponds to "helicity", that is spin along the direction of motion. It is quantized in half-integers, like ordinary spin.

Famous examples of the peculiarities of helicity, for the classification of particle states, are Weyl ("two component") neutrinos, which have helicity $-\frac{1}{2}$ only, photons, which have helicity ± 1 only, and gravitons, which have helicity ± 2 only. (The appearance of equal and opposite spins, for particles which are their own antiparticles, is connected with the requirements of quantum field theory, closely related to the CPT theorem.)

We no longer think that neutrinos are $m = 0$, but the general notion of Weyl fermions, i.e. that one has particles – or, in practice, underlying fields – with just one handedness is crucial to our formulation of electroweak symmetry.

The absence of coupling for helicity 0 “longitudinal” photons is the soul of gauge invariance. The electromagnetic 4-potential A looks like a vector field. As such it does contain longitudinal pieces, but gauge invariance insures they don't couple, and therefore effectively don't exist. Similarly, the absence of helicity $0, \pm 1$ from the metric field is the soul of general covariance.

In the Higgs mechanism, when massless particles “acquire mass”, they need to organize themselves into ordinary spin representations, from scattered helicity bits. The physical particles generally embody pieces from several different underlying fields in the more fundamental, pre-symmetry breaking description.