

# Algebra of Majorana Doubling

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## Abstract

Motivated by the problem of identifying Majorana mode operators at junctions, we analyze a basic algebraic structure leading to a doubled spectrum. In general the emergent mode creation operator is highly non-linear in the original effective mode operators, and therefore also in the underlying electron creation and destruction operators. We briefly compare and contrast related issues in the Pfaffian quantum Hall state.

## *Introduction*

The existence of Majorana modes in condensed matter systems [1–6] is intrinsically interesting, in that it embodies a qualitatively new and deeply quantum mechanical phenomenon [7, 8]. It is also possible that such modes might have useful applications, particularly in quantum information processing [9, 10].

Let us briefly recall the simplest, yet representative, model for such modes, Kitaev’s wire segment [11]. We imagine  $N$  ordered sites are available to our electrons, so we have creation and destruction operators  $a_j^\dagger, a_k$ ,  $1 \leq j, k \leq N$ , with  $\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0$  and  $\{a_j^\dagger, a_k\} = \delta_{jk}$ . The same commutation relations can be expressed using the hermitean and antihermitean parts of the  $a_j$ , leading to a Clifford algebra, as follows:

$$\begin{aligned}
 \gamma_{2j-1} &= a_j + a_j^\dagger \\
 \gamma_{2j} &= \frac{a_j - a_j^\dagger}{i} \\
 \{\gamma_k, \gamma_l\} &= 2\delta_{kl}
 \end{aligned}
 \tag{1}$$

Now let us compare the Hamiltonians

$$H_0 = -i \sum_{j=1}^N \gamma_{2j-1} \gamma_{2j} \quad (2)$$

$$H_1 = -i \sum_{j=1}^{N-1} \gamma_{2j} \gamma_{2j+1} \quad (3)$$

Since  $-i\gamma_{2j-1}\gamma_{2j} = 2a_j^\dagger a_j - 1$ ,  $H_0$  simply measures the total occupancy. It is a normal, if unusually trivial, electron Hamiltonian.

$H_1$  strongly resembles  $H_0$  but there are three major differences. One difference emerges, if we re-express  $H_1$  in terms of the  $a_j$ . We find that it is local in terms of those variables, in the sense that only neighboring sites are connected, but that in addition to conventional hopping terms of the type  $a_j a_{j+1}^\dagger$  we have terms of the type  $a_j a_{j+1}$ , and their hermitean conjugates. The  $aa$  type, which we may call superconductive hopping, does not conserve electron number, and is characteristic of a superconducting (pairing) state. A second difference grows out of a similarity: since the algebra Eqn. (1) of the  $\gamma_j$  is uniform in  $j$ , we can interpret the products  $\gamma_{2j}\gamma_{2j+1}$  that appear in  $H_1$  in the same fashion that we interpret the products  $\gamma_{2j-1}\gamma_{2j}$  that appear in  $H_0$ , that is as occupancy numbers. The effective fermions that appear in these numbers, however, are not the original electrons, but mixtures of electrons and holes on neighboring sites.

The third and most profound difference is that the operators  $\gamma_1, \gamma_{2N}$  do not appear at all in  $H_1$ . These are the Majorana mode operators. They commute with the Hamiltonian, square to the identity, and anticommute with each other. The action of  $\gamma_1$  and  $\gamma_{2N}$  on the ground state implies a degeneracy of that state, and the corresponding modes have zero energy. Kitaev [11] shows that similar behavior occurs for a family of Hamiltonians allowing continuous variation of microscopic parameters, i.e. for a universality class. Within that universality class one has hermitean operators  $b_L, b_R$  on the two ends of the wire whose action is exponentially (in  $N$ ) localized and commute with the Hamiltonian up to exponentially small corrections, that satisfy the characteristic relations  $b_L^2 = b_R^2 = 1$ . In principle there is a correction Hamiltonian

$$H_c \propto -ib_L b_R \quad (4)$$

that will encourage us to re-assemble  $b_L, b_R$  into an effective fermion creation-destruction pair, and realize  $H_c$  as its occupation number. But for a long wire and weak interactions we expect the coefficient of  $H_c$  to be very small,

since the modes excited by  $b_L, b_R$  are spatially distant, and for most physical purposes it will be more appropriate to work with the local variables  $b_L, b_R$ .

### *Algebraic Structure*

The following considerations will appear more pointed if we explain their origin in the following little puzzle. Let us imagine we bring together the ends of three wires supporting Majorana modes  $b_1, b_2, b_3$ . Thus we have the algebra

$$\{b_j, b_k\} = 2\delta_{jk} \quad (5)$$

The  $b_j$  do not appear in their separate wire Hamiltonians, but we can expect to have interactions

$$H_{\text{int.}} = -i(\alpha b_1 b_2 + \beta b_2 b_3 + \gamma b_3 b_1) \quad (6)$$

which plausibly arise from normal or superconductive inter-wire electron hopping. We assume here that the only important couplings among the wires involve the Majorana modes. This is appropriate if the remaining modes are gapped and the interaction is weak – for example, if we only include effects of resonant tunneling. We shall relax this assumption in due course.

We might expect, heuristically, that the interactions cause two Majorana degrees of freedom to pair up to form a conventional fermion degree of freedom, leaving one Majorana mode behind.

On the other hand, the algebra in Eqn. (5) can be realized using Pauli  $\sigma$  matrices, in the form  $b_j = \sigma_j$ . In that realization, we have simply  $H = \alpha\sigma_3 + \beta\sigma_1 + \gamma\sigma_2$ . But that Hamiltonian has eigenvalues  $\pm\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}$ , with neither degeneracy nor zero mode. In fact a similar problem arises even for “junctions” containing a single wire, since we could use  $b_R = \sigma_1$  (and  $b_L = \sigma_2$ ).

The point is that the algebra of Eqn. (5) is conceptually incomplete. It does not incorporate relevant implications of electron number parity, or in other words electron number modulo two. For the operator

$$P \equiv (-1)^{N_e} \quad (7)$$

that implements electron number parity we should have

$$P^2 = 1 \quad (8)$$

$$[P, H_{\text{eff.}}] = 0 \quad (9)$$

$$\{P, b_j\} = 0 \quad (10)$$

Eqn. (8) follows directly from the motivating definition. Eqn. (9) reflects the fundamental constraint that electron number modulo two is conserved in the theories under consideration, and indeed under very broad – possibly universal – conditions. Eqn. (10) reflects, in the context of [11], that the  $b_j$  are linear functions of the  $a_k, a_l^\dagger$ , but is more general. Indeed, it will persist under any “dressing” of the  $b_j$  operators induced by interactions that conserve  $P$ . Below we will see striking examples of this persistence.

The preceding puzzle can now be addressed. Including the algebra of electron parity operator, we take a concrete realization of operators as  $b_1 = \sigma_1 \otimes I$ ,  $b_2 = \sigma_3 \otimes I$ ,  $b_3 = \sigma_2 \otimes \sigma_1$  and  $P = \sigma_2 \otimes \sigma_3$ . This choice represents the algebra Eqns. (5, 8-10). The Hamiltonian represented in this enlarged space contains doublets at each energy level. (Related algebraic structures are implicit in [12]. See also [13–16] for more intricate, but model-dependent, constructions.)

### *Emergent Majorana Modes*

Returning to the abstract analysis, consider the special operator

$$\Gamma \equiv -ib_1b_2b_3 \tag{11}$$

It satisfies

$$\Gamma^2 = 1 \tag{12}$$

$$[\Gamma, b_j] = 0 \tag{13}$$

$$[\Gamma, H_{\text{eff.}}] = 0 \tag{14}$$

$$\{\Gamma, P\} = 0 \tag{15}$$

Eqns. (12, 13) follow directly from the definition, while Eqn. (14) follows, given Eqn. (13), from the requirement that  $H_{\text{eff.}}$  should contain only terms of even degree in the  $b$ s. That requirement, in turn, follows from the restriction of the Hamiltonian to terms even under  $P$ . Finally Eqn. (15) is a direct consequence of Eqn.(10) and the definition of  $\Gamma$ .

This emergent  $\Gamma$  has the characteristic properties of a Majorana mode operator: It is hermitean, squares to one, and has odd electron number parity. Most crucially, it commutes with the Hamiltonian, but is not a function of the Hamiltonian. We can highlight the relevant structure by going to a basis where  $H$  and  $P$  are both diagonal. Then from Eqn. (15), we see that  $\Gamma$  takes states with  $P = \pm 1$  into states of the same energy with  $P = \mp 1$ . This doubling applies to all energy eigenstates, not only the

ground state. It is reminiscent of, but differs from, Kramers doubling. (No antiunitary operation appears, nor is  $T$  symmetry assumed.)

One also has a linear operator

$$w \equiv \alpha b_3 + \beta b_1 + \gamma b_2 \quad (16)$$

that commutes with the Hamiltonian. However it is not independent of  $\Gamma$ , since we have

$$w = H\Gamma \quad (17)$$

The same considerations apply to a junction supporting any odd number  $p$  of Majorana mode operators, with

$$\Gamma \equiv i^{\frac{p(p-1)}{2}} \prod_{j=1}^p \gamma_j \quad (18)$$

For even  $p$ , however, we get a commutator instead of an anticommutator in Eqn.(15), and the doubling construction fails. For odd  $p \geq 5$  generally there is no linear operator, analogous to the  $w$  of Eqn. (16), that commutes with  $H$ .

Now let us revisit the approximation of keeping only the interactions of the Majorana modes from the separate wires. We can in fact, without difficulty, include any finite number of “ordinary” creation-annihilation modes from each wire, thus including all degrees of freedom that overlap significantly with the junction under consideration. These can be analyzed, as in Eqn. (1), into an even number of additional  $\gamma$  operators, to include with the odd number of  $b_j$ . But then the product  $\Gamma'$  of all these operators, including both types (and the appropriate power of  $i$ ), retains the good properties Eqn. (12) of the  $\Gamma$  operator we had before.

### *Pfaffian Vortices*

It is interesting to compare the answer to a similar question in another physical context where Majorana modes arise [4], that is fractional quantum Hall effects of the Pfaffian type. Following the notation and framework of [17], appropriate wave functions for the state with four quasi-particles at positions  $a, b, c, d$  can be constructed in the form

$$\begin{aligned} \Psi_1(z_j, a, b, c, d) &= \text{Pf} \frac{(z_j - a)(z_j - b)(z_k - c)(z_k - d) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \\ \Psi_2(z_j, a, b, c, d) &= \text{Pf} \frac{(z_j - a)(z_j - c)(z_k - b)(z_k - d) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \end{aligned}$$

$$\Psi_3(z_j, a, b, c, d) = \text{Pf} \frac{(z_j - a)(z_j - d)(z_k - b)(z_k - c) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \quad (19)$$

where Pf denotes the Pfaffian and  $\Psi_0$  contains the standard Laughlin-Landau factors for filling fraction  $1/2$ . Within the Pfaffian each quasi-particle acts on one member of a pair, and in each of  $\Psi_1, \Psi_2, \Psi_3$  the quasi-particles themselves are paired off, so that each quasiparticle act on the same electrons as its mate. In  $\Psi_1$   $ab$  and  $cd$  are paired in this sense, and so forth. It can be shown, by direct calculation, that  $\Psi_1, \Psi_2, \Psi_3$  do not represent three independent states, since there is an  $(a, b, c, d$ -dependent) linear relation among them. There remain 2 physical states. This is the number required by a *minimal* implementation of the nonabelian statistics, which can be based on the Clifford algebra with four generators [18].

Now formally we can take one of the quasi-particles off to infinity, and arrive at corresponding wave functions for three quasiparticles

$$\begin{aligned} \tilde{\Psi}_1(z_j, a, b, c) &= \text{Pf} \frac{(z_j - a)(z_j - b)(z_k - c) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \\ \tilde{\Psi}_2(z_j, a, b, c) &= \text{Pf} \frac{(z_j - a)(z_j - c)(z_k - b) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \\ \tilde{\Psi}_3(z_j, a, b, c) &= \text{Pf} \frac{(z_j - a)(z_k - b)(z_k - c) + (j \leftrightarrow k)}{z_j - z_k} \Psi_0(z_j) \end{aligned} \quad (20)$$

(It is also possible to use wave functions based on the “center-of-mass” Pfaffians  $\text{Pf} \frac{(z_j - a)(z_k - b)(\frac{z_j + z_k}{2} - c) + (j \leftrightarrow k)}{z_j - z_k}$ , since either set can be expressed in terms of the other.) We find that there is no further reduction; there is still a two-dimensional space of states spanned by these wave functions, as required for a minimal (non-trivial) representation of the Clifford algebra with three generators. In this context, then, it appears that the minimal spinor representation always suffices; no analogue of the electron parity operator is implemented.

*Comments*

1. The algebraic structure defined by Eqns. (8-10) is fully non-perturbative. It may be taken as the definition of the universality class supporting Majorana modes. The construction of  $\Gamma$  (in its most general form) and its consequences Eqns. (12-15) reproduces that structure, allowing for additional interactions, with  $\Gamma$  playing the role of an emergent  $b$ . The definition of  $\Gamma$ , the consequences Eqns. (12-15), and the deduction of doubling are likewise fully non-perturbative.

2. If we have a circuit with several junctions  $j$ , the emergent  $\Gamma_j$  will obey the Clifford algebra

$$\{\Gamma_j, \Gamma_k\} = 2\delta_{jk} \quad (21)$$

This applies also to junctions with  $p = 1$ , i.e. simple terminals; nor need the circuit be connected.

3.  $\Gamma$  is at the opposite extreme from a single-particle operator. The corresponding mode is associated with the *product* wave function over the modes associated with the  $b_j$ . In this sense we have extreme valence-bond (Heitler-London) as opposed to linear (Mulliken) orbitals. The contrast is especially marked, of course, for large  $p$ .

4. A Clifford algebra on an even number of generators that commute with the Hamiltonian can be re-organized, by inverting the procedure of Eqn. (1), into a supersymmetry algebra. Thus our constructions support an emergent supersymmetry. This supersymmetry algebra commutes with the Hamiltonian, but does not contain it. (Compare [19], where an emergent supersymmetry, relying on  $T$  symmetry, has been discussed in the context of Majorana modes.)

5. One can modify the preceding construction by using, in place of the  $\Gamma_j$  matrices, matrices of the type

$$\tilde{\Gamma}_j \propto \sqrt{H} \Gamma_j \quad (22)$$

to achieve a closed supersymmetry algebra, now including the Hamiltonian in the anticommutators. One could also consider more elaborate construction, in which pieces of the total Hamiltonian are assigned to different  $\Gamma_j$ , exploiting locality conditions among the underlying  $a$  operators to insure appropriate anticommutators. Of course the  $\sqrt{H}$  operators themselves will not be local, except for specially crafted  $H$ .

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