Majorana-ism

Photons are their own antiparticles. They are described by real-number fields, which simply change sign under charge conjugation. When Dirac introduced his equation for electrons, it seemed to involve complex numbers intrinsically. The interpretation of the Dirac equation was problematic, initially, but when that interpretation matured, that appearance of complex numbers seemed to be a profound, advantageous feature. It was tied up both with the fact that the electron is charged, and therefore subject to the phase rotations associated with electromagnetic gauge invariance, and with the existence of positrons. Indeed the electron field destroys electrons and creates positrons, while its complex (or more accurately, Hermitean) conjugate does the opposite.

After Pauli suggested the existence of electrically neutral neutrinos, and Fermi made them the basis of an impressive, quantitative theory of beta decay, it became interesting to reconsider, whether one could have spin $\frac{1}{2}$ particles that are their own antiparticles. Could one, specifically, have a version of the Dirac equation that involved real fields? This was a mathematical question asked, and answered, by Majorana.

Hardly anything was known about neutrinos in those days; now we know a lot. But the issue Majorana's work, when suitably interpreted, raises, defines a central remaining physical question concerning neutrinos.

Majorana Basis

In order that the Dirac-type equation

$$(i\gamma^\mu \partial_\mu + m)\psi \ = \ 0 \tag{1}$$

support solutions with a real field ψ, we must have that the γ matrices are pure imaginary. We want, of course, that they should satisfy the Clifford-Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} \ = \ 2\,\eta^{\mu\nu} \tag{2}$$

and that γ^0 is Hermitean while the spatial γ^j are anti-Hermitean. Here is a purely imaginary solution to those requirements:

$$\begin{aligned}
\gamma^0 &= \sigma_2 \otimes \sigma_3 \\
\gamma^1 &= i\sigma_1 \otimes 1 \\
\gamma^2 &= i\sigma_2 \otimes \sigma_2 \\
\gamma^3 &= i\sigma_3 \otimes 1
\end{aligned} \tag{3}$$

I will call a set of γ^μ-matrices that are purely imaginary, a Majorana basis.

Chirality and Charge Conjugation

The γ_5 matrix used in defining chiral projections is

$$\gamma_5 \;\equiv\; i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4}$$

is also pure imaginary in a Majorana basis. The i is not negotiable, since we must require $(\gamma_5)^2 = 1$. Only then will the projection operators

$$\Pi_\pm \;\equiv\; \frac{1 \pm \gamma_5}{2} \tag{5}$$

satisfy the basic requirement $\Pi_\pm^2 = \Pi_\pm$.

Since left-handed fermion fields, by definition, satisfy

$$\begin{aligned} \Pi_+ \, \psi_L &= 0 \\ \Pi_- \, \psi_L &= \psi_L \end{aligned} \tag{6}$$

and the $\Pi_\pm$, as we have just seen, are intrinsically complex, we cannot have non-trivial real chiral fermion fields.

Of course, we can use a Majorana basis even for complex ψ fields. The special virtue of such a basis is that it makes the charge conjugation operation especially transparent. Indeed if ψ satisfies the free Dirac equation, then so does ψ^*, so we have a symmetry operation of the simple form

$$\psi' \to \psi^* \tag{7}$$

For the interacting electromagnetic Dirac equation, we must also reverse the sign of A_μ.

In the basic quantization of the electron field, we face the issue of choosing a preferred "vacuum" state. The standard choice is to choose the charge-conjugation invariant state of minimal energy. This formulation makes the fictitious nature of the Dirac sea, in this context, clear. In condensed matter contexts typically we must specify a chemical potential, and there is no fundamental symmetry between particles and holes.

Note that real ("Majorana") fields can be invariant, or get multiplied by definite numbers, under transformations of the kind Eqn. (7). In that sense Majorana fields represent particles with definite charge conjugation, i.e. particles that are their own antiparticles.

On the other hand[1], since

$$\Pi_\pm^* \;=\; \Pi_\mp \tag{8}$$

chiral fields transform, by complex conjugation, into fields of the opposite chirality.

[1] Rueful smile here.

Quasi-Majorana Pseudo-Chiral Neutrinos

Among the most basic observations on neutrinos are

1. The conservation of separate lepton numbers L_e, L_μ, L_τ in most reactions and decays.
2. Neutrinos emitted in decays or reactions have left-handed chirality, while antineutrinos have right-handed chirality.
3. The violation of these separate quantum numbers when neutrinos are allowed to travel over large distances (neutrino oscillations).

Neutrino oscillations provide evidence for mass terms that are not diagonal with respect to the separate lepton numbers, though as yet no observation has revealed violation of the total $L_e + L_\mu + L_\tau$. Mass terms, diagonal or not, are incompatible with chiral projections. Thus the familiar "left-handed neutrino" which particle physicists thought for decades that they'd been dealing with can only be an approximation. It must have some admixture of right-handed chirality.

Thus a fundamental question arises: Are these right-handed components of neutrinos something entirely new – or could they involve the same degrees of freedom we met before, in antineutrinos?

At first hearing that question might sound quite strange, since neutrinos and anti-neutrinos have quite different properties. How could there be a piece of the neutrino, that acts like an antineutrino? But of course if the piece is small enough, it might be compatible with observations. And if the energy of our neutrinos is large compared to their mass, the admixture of opposite chirality will be small. Indeed, it is proportional to m/E. In the phenomenology of neutrino oscillations, and taking into account cosmological constraints, we are led to masses $m < eV$, and so in most practical experiments m/E is a very small parameter.

A more vivid way of posing the question: Are neutrinos and antineutrinos the same particles, just observed in different states of motion? The observed distinctions might just represent unusual spin-dependent (or more properly helicity-dependent) interactions.

These questions are usually posed in the cryptic form: Are neutrinos Majorana particles?

Majorana Mass

To put our questions on a firm mathematical basis, the central issue is to formulate how we can describe a massive spin-$\frac{1}{2}$ particle using just two (not four) degrees of freedom. We want the antiparticle to involve the same degrees of freedom as the particle. Concretely, we want to investigate how the hypothesis

$$\psi_R \overset{?}{=} \psi_L^* \tag{9}$$

(in a Majorana basis) might be compatible with non-zero mass. Applying a chiral projection to the Dirac equation in general gives us the form

$$i\gamma^\mu \partial_\mu \psi_L + M\psi_R \ = \ 0 \tag{10}$$

and so we are led to contemplate

$$i\gamma^\mu \partial_\mu \psi_L + M\psi_L^* \ = \ 0 \tag{11}$$

(Mathematical/historical aside: If Eqn. (9) holds, we can derive both ψ_L and ψ_R by projection from a single four-component *real* field, i.e.

$$\psi \ \equiv \ \psi_L + \psi_R \ = \ \psi_L + \psi_L^* \tag{12}$$

This is the link to Majorana's original idea.)

The appearance of Eqn. (11) is unusual, and we may wonder how it could arise as a field equation, from a Lagrangian density. Usually we consider mass terms

$$\mathcal{L}_M \ \propto \ \bar{\psi}\psi \ = \ \psi^\dagger \gamma_0 \psi \tag{13}$$

Now if we write everything in terms of ψ_L, using Eqn. (9), we find

$$\psi^\dagger \gamma_0 \psi \ \rightarrow \ (\psi_L)^T \gamma_0 \psi_L + (\psi_L^*)^T \gamma_0 \psi_L^* \tag{14}$$

where ^{T} denotes transpose. In verifying that these terms are non-trivial, whereas the remaining cross-terms vanish, it is important to note that γ_5 is antisymmetric, i.e., that it changes sign under transpose. That is true, because γ_5 is both Hermitean and pure imaginary. Varying this form, together with the conventional kinetic term

$$\mathcal{L} \ \propto \ (\psi_L^*)^T \, i\gamma^\mu \partial_\mu \psi_L + h.c. \tag{15}$$

will give us Eqn. (11).

From any of these perspectives: the basic formation Eqn. (9), the Dirac equation, or the underlying Lagrangian, we see that the Majorana fermion hypothesis is not compatible with a phase symmetry for ψ_L. In the context of the standard model, we cannot support mass terms that are simply quadratic in fermion fields of the same chirality, as in Eqn. (14), because they will never be invariant under hypercharge symmetry.

On the other hand if we have a "right-handed neutrino" N_R that is invariant under all the gauge symmetries of the standard model, there is nothing to forbid it having a (Majorana) mass, even if there is no independent N_L.

If we relax the requirement of renormalizability slightly, to allow mass dimension 5 interactions, we can get a (Majorana) mass for the neutrino, without introducing new degrees of freedom. Indeed, we can have contributions of the form

$$\mathcal{L}_{\text{neutrino mass}} \ \propto \ L_L^a \phi_a^* \, L_L^b \phi_b^* + h.c. \tag{16}$$

where, as always, L is the left-handed lepton doublet, and ϕ the Higgs doublet. When we expand ϕ around its vacuum expectation value, we get a mass term of the form Eqn. (14) for the ordinary neutrino, $\phi_L \rightarrow \nu_L$.

Next Steps

1. I have written the equations suppressing flavor dependence. That can be put back in, straightforwardly. Worthy of remark is that since the Majorana mass term

$$\psi_L^{\rho f}\,(\gamma_0)_{\rho\sigma}\,M_{fg}\,\psi_L^{\sigma g} \tag{17}$$

where ρ, σ are spinor indices and f, g flavor indices, must be overall antisymmetric in exchange of the fields, and γ_0 is antisymmetric, we must take M_{fg} symmetric.

2. It is often advantageous, in constructing candidate unified theories with (spontaneously broken) symmetries larger than the standard model, to include an N_R as described above. It makes the lepton sector visibly parallel to the quark sector.

3. Because N_R can have an "intrinsic" mass, independent of electroweak symmetry breaking, it is not unreasonable to contemplate that its mass might be very large. Other considerations, specifically the unification of running couplings in unified theories, strongly endorse this possibility.

4. Normal (invariant) Yukawa couplings of L_L to N_R through the Higgs field, might be expected to give us massive neutrinos, with masses similar to the masses of quarks and charged leptons. However if N_R has its own large intrinsic mass, the the effect of these "ordinary" mass couplings will be to connect us to very high-energy degrees of freedom (i.e. N_R). We get an effect involving only light degrees of freedom in second-order perturbation theory, with M_N appearing as an energy denominator. This is the essence of the so-called see-saw mechanism for generating neutrino masses. It can explain, semi-quantitatively, why the observed masses of neutrinos are so very much smaller than those of quarks and charged leptons.

5. The most direct test of the Majorana neutrino hypothesis, would be to see if pairs of neutrinos, brought to rest, annihilate one another. Unfortunately, although we are presumably immersed in a cosmological neutrino background, similar to the microwave photon background, and these neutrinos should have cooled to near rest, no one has designed a practical experiment to probe this issue. The only experiments on the drawing board, which get at these issues, are searches for neutrinoless double beta decay, as a sign of total lepton number violation.

Condensed Matter: Superconductivity and Majorinos

In the context of superconductivity, Majorana-type mass terms arise very naturally. Indeed, starting with a four-electron interaction

$$\mathcal{L} \;\sim\; \bar\psi\bar\psi\psi\psi \tag{18}$$

and allowing the formation of a condensate $\Delta \sim \langle\psi\psi\rangle$, we get terms of the form

$$\mathcal{L} \rightarrow \Delta^*\psi\psi + \Delta\psi^*\psi^* \tag{19}$$

An exciting frontier of current investigation in condensed matter physics and superconductivity involves a very special kind of Majorana fermion, that is associated to special points in space and has zero mass. Since, regarded as a particle, it is both exceedingly limited and exceedingly light, it is naturally to refer to this kind of fermion with the diminutive "Majorino". For an introduction to these ideas, see the accompanying paper, which I've attached as a very long appendix, and the references therein.

MIT-CTP-4481

Algebra of Majorana doubling

Jaehoon Lee and Frank Wilczek
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge MA 02139 USA
(Dated: November 6, 2013)

Motivated by the problem of identifying Majorana mode operators at junctions, we analyze a basic algebraic structure leading to a doubled spectrum. For general (nonlinear) interactions the emergent mode creation operator is highly non-linear in the original effective mode operators, and therefore also in the underlying electron creation and destruction operators. This phenomenon could open up new possibilities for controlled dynamical manipulation of the modes. We briefly compare and contrast related issues in the Pfaffian quantum Hall state.

PACS numbers: 74.20.Mn, 02.10.De, 74.81.Fa, 03.65.Vf

Introduction: The existence of Majorana modes in condensed matter systems [1–6] is intrinsically interesting, in that it embodies a qualitatively new and deeply quantum mechanical phenomenon [7, 8]. It is also possible that such modes might have useful applications, particularly in quantum information processing [9, 10]. One feature that makes Majorana modes useful is that they generate a doubled spectrum. Repeated doubling generates a huge Hilbert space of degenerate states, which is the starting point for possible quantum computational applications. In this letter, we explore the precise algebraic structure underlying that degeneracy.

Consider the situation where multiple Majorana modes come together to form a junction, as might occur in a network of superconducting wires that support a non-trivial topological phase. Several experimental groups are developing physical embodiments of Majorana modes, for eventual use in such complex quantum circuits. (For a useful sampling of very recent activity, see the collection of abstracts from the July 12-18 2013 Erice workshop : [11].) We consider a fundamental issue that arises in analyzing such circuits. For each odd junction of a circuit, we identify a remarkably simple, explicit non-linear operator Γ that implements the doubling. We point out interesting algebraic properties of Γ, and emphasize its tight connection with fermion parity. We find these results, in their power and simplicity, encouraging for further developments of technology using Majorana wire circuits. In particular it should be possible, by incorporating the effect a very broad class of interactions systematically, to put the analysis of mode transport through tri-junctions and Josephson couplings on a more general and rigorous footing.

Review of Kitaev's Wire Model: Let us briefly recall the simplest, yet representative, model for such modes, Kitaev's wire segment [12]. We imagine N ordered sites are available to our electrons, so we have creation and destruction operators $a_j^\dagger, a_k$, $1 \leq j,k \leq N$, with $\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0$ and $\{a_j^\dagger, a_k\} = \delta_{jk}$. The same commutation relations can be expressed using the hermitean and antihermitean parts of the a_j, leading to a Clifford algebra, as follows:

$$\begin{aligned} \gamma_{2j-1} &= a_j + a_j^\dagger \\ \gamma_{2j} &= \frac{a_j - a_j^\dagger}{i} \\ \{\gamma_k, \gamma_l\} &= 2\,\delta_{kl}. \end{aligned} \tag{1}$$

Now let us compare the Hamiltonians

$$H_0 = -i \sum_{j=1}^{N} \gamma_{2j-1}\gamma_{2j} \tag{2}$$

$$H_1 = -i \sum_{j=1}^{N-1} \gamma_{2j}\gamma_{2j+1}. \tag{3}$$

Since $-i\gamma_{2j-1}\gamma_{2j} = 2a_j^\dagger a_j - 1$, H_0 simply measures the total occupancy. It is a normal, if unusually trivial, electron Hamiltonian.

H_1 strongly resembles H_0 but there are three major differences. One difference emerges, if we re-express H_1 in terms of the a_j. We find that it is local in terms of those variables, in the sense that only neighboring sites are connected, but that in addition to conventional hopping terms of the type $a_j a_{j+1}^\dagger$ we have terms of the type $a_j a_{j+1}$, and their hermitean conjugates. The aa type, which we may call superconductive hopping, does not conserve electron number, and is characteristic of a superconducting (pairing) state. A second difference grows out of a similarity: since the algebra Eqn. (1) of the γ_j is uniform in j, we can interpret the products $\gamma_{2j}\gamma_{2j+1}$ that appear in H_1 in the same fashion that we interpret the products $\gamma_{2j-1}\gamma_{2j}$ that appear in H_0, that is as occupancy numbers. The effective fermions that appear in these numbers, however, are not the original electrons, but mixtures of electrons and holes on neighboring sites.

The third and most profound difference is that the operators γ_1, γ_{2N} do not appear at all in H_1. These are the Majorana mode operators. They commute with the Hamiltonian, square to the identity, and anticommute with each other. The action of γ_1 and γ_{2N} on the ground state implies a degeneracy of that state, and the corresponding modes have zero energy. Kitaev [12] shows that similar behavior occurs for a family of Hamiltonians allowing continuous variation of microscopic parameters, i.e. for a universality class. Within that universality class one has hermitean operators b_L, b_R on the two ends of the wire whose action is exponentially (in N) localized and commute with the Hamiltonian up to exponentially small corrections, that satisfy the characteristic relations $b_L^2 = b_R^2 = 1$. In principle there is a correction Hamiltonian,

$$H_c \;\propto\; -ib_L b_R, \tag{4}$$

that will encourage us to re-assemble b_L, b_R into an effective fermion creation-destruction pair, and realize H_c as its occupation number. But for a long wire and weak interactions we expect the coefficient of H_c to be very small, since the modes excited by b_L, b_R are spatially distant, and for most physical purposes it will be more appropriate to work with the local variables b_L, b_R.

Algebraic Structure: The following considerations will appear more pointed if we explain their origin in the following little puzzle. Let us imagine we bring together the ends of three wires supporting Majorana modes b_1, b_2, b_3. Thus we have the algebra

$$\{b_j, b_k\} \;=\; 2\delta_{jk}. \tag{5}$$

The b_j do not appear in their separate wire Hamiltonians, but we can expect to have interactions

$$H_{\text{int.}} \;=\; -i(\alpha\, b_1 b_2 + \beta\, b_2 b_3 + \gamma\, b_3 b_1) \tag{6}$$

which plausibly arise from normal or superconductive inter-wire electron hopping. We assume here that the only important couplings among the wires involve the Majorana modes. This is appropriate if the remaining modes are gapped and the interaction is weak – for example, if we only include effects of resonant tunneling. We shall relax this assumption in due course.

We might expect, heuristically, that the interactions cause two Majorana degrees of freedom to pair up to form a conventional fermion degree of freedom, leaving one Majorana mode behind.

On the other hand, the algebra in Eqn. (5) can be realized using Pauli σ matrices, in the form $b_j = \sigma_j$. In that realization, we have simply $H = \alpha\sigma_3 + \beta\,\sigma_1 + \gamma\,\sigma_2$. But that Hamiltonian has eigenvalues $\pm\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}$, with neither degeneracy nor zero mode. In fact a similar problem arises even for "junctions" containing a single wire, since we could use $b_R = \sigma_1$ (and $b_L = \sigma_2$).

The point is that the algebra of Eqn. (5) is conceptually incomplete. It does not incorporate relevant implications of electron number parity, or in other words electron number modulo two. For the operator

$$P \;\equiv\; (-1)^{N_e} \tag{7}$$

that implements electron number parity we should have

$$\begin{aligned} P^2 \;&=\; 1 &(8)\\ [P, H_{\text{eff.}}] \;&=\; 0 &(9)\\ \{P, b_j\} \;&=\; 0. &(10)\end{aligned}$$

Eqn. (8) follows directly from the motivating definition. Eqn. (9) reflects the fundamental constraint that electron number modulo two is conserved in the theories under consideration, and indeed under very broad – possibly universal – conditions. Eqn. (10) reflects, in the context of [12], that the b_j are linear functions of the $a_k, a_l^\dagger$, but is more general. Indeed, it will persist under any "dressing" of the b_j operators induced by interactions that conserve P. Below we will see striking examples of this persistence.

The preceding puzzle can now be addressed. Including the algebra of electron parity operator, we take a concrete realization of operators as $b_1 = \sigma_1 \otimes I$, $b_2 = \sigma_3 \otimes I$, $b_3 = \sigma_2 \otimes \sigma_1$ and $P = \sigma_2 \otimes \sigma_3$. This choice represents the algebra Eqns. (5, 8-10). The Hamiltonian represented in this enlarged space contains doublets at each energy level. (Related algebraic structures are implicit in [13]. See also [14–17] for more intricate, but model-dependent, constructions.)

Emergent Majorana Modes: Returning to the abstract analysis, consider the special operator

$$\Gamma \;\equiv\; -ib_1 b_2 b_3. \tag{11}$$

It satisfies

$$\begin{aligned} \Gamma^2 \;&=\; 1 &(12)\\ [\Gamma, b_j] \;&=\; 0 &(13)\\ [\Gamma, H_{\text{eff.}}] \;&=\; 0 &(14)\\ \{\Gamma, P\} \;&=\; 0. &(15)\end{aligned}$$

Eqns. (12, 13) follow directly from the definition, while Eqn. (14) follows, given Eqn. (13), from the requirement that $H_{\text{eff.}}$ should contain only terms of even degree in the b_is. That requirement, in turn, follows from the restriction of the Hamiltonian to terms even under P. Finally Eqn. (15) is a direct consequence of Eqn.(10) and the definition of Γ.

This emergent Γ has the characteristic properties of a Majorana mode operator: It is hermitean, squares to one, and has odd electron number parity. Most crucially, it commutes with the Hamiltonian, but is not a function of the Hamiltonian. We can highlight the relevant structure by going to a basis where H and P are both diagonal. Then from Eqn. (15), we see that Γ takes states with $P = \pm 1$ into states of the same energy with $P = \mp 1$. This doubling applies to all energy eigenstates, not only the ground state. It is reminiscent of, but differs from, Kramers doubling. (No antiunitary operation appears, nor is T symmetry assumed.)

One also has a linear operator

$$w \;\equiv\; \alpha\, b_3 + \beta\, b_1 + \gamma\, b_2 \tag{16}$$

that commutes with the Hamiltonian. However it is not independent of Γ, since we have

$$w \;=\; H\,\Gamma. \tag{17}$$

The same considerations apply to a junction supporting any odd number p of Majorana mode operators, with

$$\Gamma \;\equiv\; i^{\frac{p(p-1)}{2}} \prod_{j=1}^{p} \gamma_j. \tag{18}$$

For even p, however, we get a commutator instead of an anticommutator in Eqn.(15), and the doubling construction fails. For odd $p \geq 5$ generally there is no linear operator, analogous to the w of Eqn. (16), that commutes with H. (If the Hamiltonian is quadratic, the existence of a linear zero mode follows from simple linear algebra – namely, the existence of a zero eigenvalue of an odd-dimensional antisymmetric matrix, as discussed in earlier analyses. But for more complex, realistic Hamiltonians, including nearby electron modes as envisaged below, that argument is insufficient, even for $p = 3$. The emergent operator Γ, on the other hand, always commutes with the Hamiltonian (Eqns. (14)), even allowing for higher order contributions such as quartic or higher polynomials in the b_is.)

Now let us revisit the approximation of keeping only the interactions of the Majorana modes from the separate wires. We can in fact, without difficulty, include any finite number of "ordinary" creation-annihilation modes from each wire, thus including all degrees of freedom that overlap significantly with the junction under consideration. These can be analyzed, as in Eqn. (1), into an even number of additional γ operators, to include with the odd number of b_j. But then the product Γ' of all these operators, including both types (and the appropriate power of i), retains the good properties Eqn. (12) of the Γ operator we had before.

Now let us briefly discuss how Γ resolves the puzzle in the previous section. If $p \geq 5$, or even at $p = 3$ with nearby electron interactions included effects, the emergent zero mode is highly non-linear entangled state involving all the wires that participate at the junction. The robustness of these conclusions results from the algebraic properties of Γ we identified.

Pfaffian Vortices: It is interesting to compare the answer to a similar question in another physical context where Majorana modes arise [4], that is fractional quantum Hall effects of the Pfaffian type. Following the notation and framework of [18], appropriate wave functions for the state with four quasi-particles at positions a, b, c, d can be constructed in the form

$$\begin{aligned}
&\Psi_1(z_j, a, b, c, d)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_j-b)(z_k-c)(z_k-d) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j)\\
&\Psi_2(z_j, a, b, c, d)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_j-c)(z_k-b)(z_k-d) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j)\\
&\Psi_3(z_j, a, b, c, d)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_j-d)(z_k-b)(z_k-c) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j)
\end{aligned} \tag{19}$$

where Pf denotes the Pfaffian and Ψ_0 contains the standard Laughlin-Landau factors for filling fraction 1/2. Within the Pfaffian each quasi-particle acts on one member of a pair, and in each of Ψ_1, Ψ_2, Ψ_3 the quasiparticles themselves are paired off, so that each quasiparticle act on the same electrons as its mate. In Ψ_1 ab and cd are paired in this sense, and so forth. It can be shown, by direct calculation, that Ψ_1, Ψ_2, Ψ_3 do not represent three independent states, since there is an (a, b, c, d-dependent) linear relation among them. There remain 2 physical states. This is the number required by a *minimal* implementation of the nonabelian statistics, which can be based on the Clifford algebra with four generators [19].

Now formally we can take one of the quasi-particles off to infinity, and arrive at corresponding wave functions for three quasiparticles [20]

$$\begin{aligned}
&\tilde{\Psi}_1(z_j, a, b, c)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_j-b)(z_k-c) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j)\\
&\tilde{\Psi}_2(z_j, a, b, c)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_j-c)(z_k-b) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j)\\
&\tilde{\Psi}_3(z_j, a, b, c)\\
&\quad = \mathrm{Pf}\,\frac{(z_j-a)(z_k-b)(z_k-c) + (j \leftrightarrow k)}{z_j - z_k}\;\Psi_0(z_j).
\end{aligned} \tag{20}$$

We find that there is no further reduction, so there is a two-dimensional space of states spanned by these wave functions, as required for a minimal (non-trivial) representation of the Clifford algebra with three generators. In this context, then, it appears that the minimal spinor representation always suffices: no analogue of the electron parity operator is implemented.

Comments :

1. The algebraic structure defined by Eqns. (8-10) is fully non-perturbative. It may be taken as the definition of the universality class supporting Majorana modes. The construction of Γ (in its most general form) and its consequences Eqns. (12-15) reproduces that structure, allowing for additional

interactions, with Γ playing the role of an emergent b. The definition of Γ, the consequences Eqns. (12-15), and the deduction of doubling are likewise fully non-perturbative.

2. If we have a circuit with several junctions j, the emergent Γ_j will obey the Clifford algebra

$$\{\Gamma_j, \Gamma_k\} \;=\; 2\delta_{jk}. \tag{21}$$

This applies also to junctions with $p = 1$, i.e. simple terminals; nor need the circuit be connected.

3. Γ is at the opposite extreme from a single-particle operator. The corresponding mode is associated with the *product* wave function over the modes associated with the b_j. In this sense we have extreme valence-bond (Heitler-London) as opposed to linear (Mulliken) orbitals. The contrast is especially marked, of course, for large p.

4. The fact that interactions modify the Majorana modes in such a simply analyzed, yet highly non-trivial fashion suggests new possibilities for circuit operations, that merit much further consideration.

5. A Clifford algebra on an even number of generators that commute with the Hamiltonian can be re-organized, by inverting the procedure of Eqn. (1), into a supersymmetry algebra. Thus our constructions support an emergent supersymmetry. This supersymmetry algebra commutes with the Hamiltonian, but does not contain it. (Compare [21], where an emergent supersymmetry, relying on T symmetry, has been discussed in the context of Majorana modes.)

6. One can modify the preceding construction by using, in place of the Γ_j matrices, matrices of the type

$$\tilde{\Gamma}_j \;\propto\; \sqrt{H}\,\Gamma_j \tag{22}$$

to achieve a closed supersymmetry algebra, now including the Hamiltonian in the anticommutators. One could also consider more elaborate construction, in which pieces of the total Hamiltonian are assigned to different Γ_j, exploiting locality conditions among the underlying a operators to insure appropriate anticommutators. Of course the $\sqrt{H}$ operators themselves will not be local, except for specially crafted H.

Acknowledgement: We thank Vincent Liu and Chetan Nayak for helpful comments. This work is supported by the U.S. Department of Energy under contract No. DE-FG02-05ER41360. J.L. is supported in part by a Samsung Scholarship.

[1] V. Mourik, K. Zou, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers and L. P. Kouwenhoven, Science **336**, 1003 (2012).
[2] R. Jackiw and P. Rossi, Nucl. Phys. B **190**, 681 (1981).
[3] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[4] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[5] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
[6] C. W. J. Beenakker, Annu. Rev. Con. Mat. Phys. **4**, 113 (2013)
[7] F. Wilczek, Nature Physics **5**, 614 (2009).
[8] F. Wilczek, Nature **486**, 195 (2012).
[9] A. Y. Kitaev, A. Ann. Phys. **321**, 2 (2006).
[10] C. Nayak, C. H. Simons, A. Stern, M. Freedman and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[11] *Conference on Majorana Physics in Condensed Matter :* `web.nano.cnr.it/mpcm13/MPCM2013_Booklet.pdf`, and references therein.
[12] A. Y. Kitaev, Phys. Usp. **44**, no. 10S, 131 (2001); arxiv:cond-mat/0010440.
[13] A. R. Akhmerov, Phys. Rev. B **82**, 020509 (2010).
[14] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nature Physics **7**, 412 (2010).
[15] B. I. Halperin, Y. Oreg, A. Stern, G. Refael, J. Alicea and F. von Oppen, Phys. Rev. B **85**, 144501 (2012).
[16] B. van Heck, A. R. Arkmerov, F. Hassler, M. Burrello and C. W. J. Beenakker, New J. Phys. **14**, 035019 (2012).
[17] D. J. Clarke, J. D. Sau, S. Tewari, Phys. Rev. B **84**, 035120 (2011).
[18] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).
[19] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[20] It is also possible to use wave functions based on the "center-of-mass" Pfaffians $\mathrm{Pf}\frac{(z_j-a)(z_k-b)(\frac{z_j+z_k}{2}-c)+(j\leftrightarrow k)}{z_j-z_k}$, since either set can be expressed in terms of the other.
[21] X. -L. Qi, T. L. Hughes, S. Raghu and S. C. Zhang, Phys. Rev. Lett. **102**, 187001 (2009).