Fermion Masses and Mixings

As we have seen, the minimal Higgs mechanism gives us a very specific account of W and Z boson properties – specifically, their masses and their relation to the underlying B and C mesons, whose couplings are fixed by gauge symmetry. This account is in good agreement with a large body of phenomenology. In particular, it relieves the deep problem of reconciling non-zero gauge boson mass with fundamental gauge symmetry. We face another deep problem, in reconciling the non-zero mass of fermions with chiral symmetry. Recall that we relied on chiral symmetry, to give us a rigorous formulation of "maximal parity violation" phenomenology, by having left-handed and right-handed helicities with different W boson interactions. But if the different helicities have different gauge transformation properties, a simple mass term will not be invariant, and so the fermions must be massless. (More pointedly: The representations of the Lorentz group that support definite chirality require zero mass.)

Fortunately, the same condensate that works to give W and Z mass can also generate masses for quarks and leptons. The theory in that sector is not nearly so elegant or unique, and it brings in many new parameters. Nevertheless it supports a predictive, and so far highly successful, phenomenology.

Quark and Charged Lepton Masses

Previously we have classified the couplings allowed by our general principles (including mass dimension ≤ 4), without regard to Higgs condensate formation. Now we will assess the consequences of that condensation. For definiteness, let us focus on term involving Q_L, U_R, and ϕ. It will be important to keep the flavor indices, too. So we are faced with

$$\mathcal{L}_M \;=\; -\,\overline{Q_L}_{\alpha a r} U_R^{\alpha s} \phi^a M_s^r \,+\, \text{h.c.} \tag{1}$$

where the Greek index is for color, the early Latin for weak isospin, and the middle Latin for flavor. "h. c." denotes Hermitean conjugate, and M_s^r is a numerical matrix. Now we substitute the condensate

$$\phi^a \;\rightarrow\; \delta^{a1}\, v \tag{2}$$

in the top component of the weak isospinor, and consolidate the notation a bit, to write

$$\mathcal{L}_M \;\rightarrow\; -\,v\overline{U}_L M U_R + \text{h.c.} \tag{3}$$

understanding that the Us are vectors in flavor space, containing the three components $(u, c, t)^T$.

The *mass matrix* vM can have both real and imaginary components. When we add in the Hermitean conjugate, we see that the imaginary components give rise to the Dirac structure

$$\bar{\psi} i \gamma_5 \psi \tag{4}$$

That is the meaning of imaginary fermion masses.

To keep the particle physics as conventional as possible, we would like to use fields whose quanta propagate as particles with definite mass and normalized energy-momentum. To accomplish this, we introduce

$$\begin{aligned} U_L &= S\tilde{U}_L \\ U_R &= T\tilde{U}_R \end{aligned} \tag{5}$$

with S and T unitary matrices. We want them to be unitary, so that the kinetic terms remain normalized (given that they were normalized before! – we'll revisit, and justify, that assumption later). In terms of these, we have

$$\mathcal{L}_M \;\rightarrow\; -v\bar{\tilde{U}}_L S^{-1} M T \tilde{U}_R + \text{h.c.} \tag{6}$$

so the effective mass matrix is

$$v\tilde{M} \;\equiv\; vS^{-1}MT \tag{7}$$

Now it is an exercise in linear algebra to show that we can choose S and T so that $\tilde{M}$ is diagonal and non-negative.

(We can reduce this to more perhaps familiar results in linear algebra, as follows. Since $M^\dagger M$ is Hermitean, we can diagonalize it, with a unitary T (this will turn out to be an appropriate name!). Thus we have

$$(MT)^\dagger(MT) \;=\; D^2 \tag{8}$$

with D diagonal. This implies that MTD^{-1} is unitary, as is

$$S \;=\; MT\frac{1}{|D|} \tag{9}$$

in an evident notation. But this is just

$$S^{-1}MT \;=\; |D| \tag{10}$$

as desired.)

We can do exactly parallel constructions for the term involving Q_L, D_R, ϕ and the term involving L_L, E_R, ϕ. They identify the masses for $(d, s, b)^T$ and $(e, \mu, \tau)^T$, respectively, and the fields that implement them, given the fundamental couplings in their original, most general form.

Cabibbo-Kobayashi-Maskawa (CKM) Matrix

We've now got the kinetic terms involving ∂ as well as the mass terms into conventional form. It is important to note, however, that we may need different matrices S_U, T_U and S_D, T_D to accomplish this feat. This enters the picture, when we look at the gauge potential part of the covariant derivatives.

The interesting terms are

$$(\overline{U}_L \overline{D}_L)\, B^a \frac{\tau^a}{2} \begin{pmatrix} U_L \\ D_L \end{pmatrix} \;=\; (\overline{\tilde{U}}_L S_U^{-1} \overline{\tilde{D}}_L S_D^{-1})\, B^a \frac{\tau^a}{2} \begin{pmatrix} S_U \tilde{U}_L \\ S_D \tilde{D}_L \end{pmatrix} \tag{11}$$

In the $a = 3$ case, where we encounter the diagonal τ_3 in isospin space, the S matrices cancel. The do not cancel, however, in the off-diagonal $a = 1, 2$ cases. Writing things out in terms of the $W^\pm$ fields that take us up or down in weak isospin, we have

$$B^1 \frac{\tau^1}{2} + B^2 \frac{\tau^2}{2} \;\equiv\; W^+ \frac{\tau^+}{\sqrt{2}} + W^- \frac{\tau^-}{\sqrt{2}} \tag{12}$$

with

$$W^\pm \;=\; \frac{B^1 \pm i B^2}{\sqrt{2}} \tag{13}$$

and we see that the effective W^+ coupling, in flavor space, involves the unitary matrix

$$S_U^{-1} S_D \;\equiv\; V \tag{14}$$

while the effective W^- coupling involves V^{-1}.

V is very close to the Cabibbo-Kobayashi-Maskawa (CKM) matrix as usually defined. To complete the job we must exploit some residual freedom in the choice of S_U, S_D. Namely, even after we have made the mass matrices diagonal and non-negative, and keeping the ∂ terms conventional (i. e., with a common, unit coefficient, diagonal in flavor space), we can do another around of redefinitions, with

$$\begin{aligned} S'_U &= T'_U = \text{diagonal phase matrix} \\ S'_D &= T'_D = \text{diagonal phase matrix} \end{aligned} \tag{15}$$

with the effect

$$V' \;=\; {S'_U}^{-1} V S'_D \tag{16}$$

This allows us to make the top row and the first column of V' real, as we have discussed previously. For two flavors, that makes the matrix entirely real, but for three flavors we are left with one non-trivial phase parameter.

The CKM matrix (for 3 flavors) involves three real parameters, and 1 phase. That is still quite a modest number of parameters, compared to the wealth of observations it successfully accounts for, notably including CP violation phenomena in both the K and B meson sectors.

Neutrino Masses and Mixings

To account for neutrino masses, without introducing new dynamical degrees of freedom, we must allow for some nonrenormalizable interactions with mass dimension 5, of the type

$$\mathcal{L}_m \;=\; -\frac{1}{\Lambda}\,\overline{L}{*_L}^{ar}\phi^*_a P_{rs} L^{bs}\phi^*_b \,+\, \text{h.c.} \tag{17}$$

Here Λ is a parameter with mass dimension 1, and P a numerical matrix. We are using a Majorana basis, so that L^*_L – a field with *right-handed* chirality – transforms as a spinor field should under Lorentz transformations. (For more background on these issues, see the preceding note "Majorana-ism".) After condensation, we have the mass term

$$\mathcal{L}_m \;\to\; -\frac{v^2}{\Lambda}\nu^r_L\gamma_0\nu^s_L P_{rs} \,+\, \text{h.c.} \tag{18}$$

In a Majorana basis γ_0 is antisymmetric, and since they are fermionic the ν^r_L anticommute, so we can take P_{rs} to be symmetric. It need not, however, be real.

Following our earlier strategy, we make the re-definition

$$\nu_L \;=\; U\tilde{\nu}_l \tag{19}$$

where U is a unitary matrix acting on the flavor indices. In the new basis we have an effective P matrix

$$\tilde{P} \;=\; U^T P U \tag{20}$$

Note that $\tilde{P}$ is symmetric if P is.

It is another exercise in linear algebra to demonstrate that we can use this sort of transformation, with an appropriate choice of U, to render $\tilde{P}$ diagonal and non-negative. Note that P contains 6 complex, or 12 real, parameters, while U contains 9 real parameters, so it is reasonable to expect we can get a canonical form for $\tilde{P}$ with $12 - 9 = 3$ parameters.

Similarly to the situation for quarks: Because the re-definitions we need to diagonalize the mass terms for the left-handed charged leptons and for the neutrinos need not agree, we will get a unitary mixing matrix in the charged current. Note, however, that we cannot make secondary phase re-definitions, analogous to Eqn. 15), on the neutrino fields. Thus in principle there are 3 residual CP violating phases. One of them is analogous to the phase in the quark case, and appears in the charged currents. It is accessible in neutrino oscillations, and there is a very active program of research aimed at measuring it. The others are only accessible through neutrino-antineutrino oscillations, which are much more demanding experimentally, and not presently practicable.