# An Example in the Exchange of Limits

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How can it *fail to be true* that for a smooth function $f(\Omega)$, the expansion of

$$h(\Omega) \;\equiv\; f(\Omega) - \Omega \frac{df}{d\Omega} \tag{1}$$

has no linear term?

To transcend that "obvious" truth, we need to expand our horizon, and bring in another variable. With

$$f(\Omega, N) = \Omega \log(1 + N\Omega) \tag{2}$$

we have

$$h \equiv f - \Omega \frac{df}{d\Omega} = \frac{-N\Omega^2}{1 + N\Omega} \tag{3}$$

Now for any finite $N$ the linear term in the expansion of $h$ in powers of $\Omega$ vanishes, but for $N \to \infty$ we have simply $h \to -\Omega$.

The mathematical theory of phase transitions, in physics, hinges on precisely this subtlety! One discovers qualitatively different behaviors for finite $N$ and for $N \to \infty$, where $N$ is the number of particles (or some other measure of the system size).